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PLANE GEOMETRY

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NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

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Entered at Stationers' Hall, London.

H.-F. PLANE GEOMETRY
W. P. 16

PREFACE

THIS book is the outgrowth of an experience of many years in the teaching of mathematics in secondary schools. The text has been used by many different teachers, in classes of all stages of development, and under varying conditions of secondary school teaching. The proofs have had the benefit of the criticisms of hundreds of experienced teachers of mathematics throughout the country. The book in its present form is therefore the combined product of experience, classroom test, and severe criticism.

The following are some of the leading features of the book:

The student is rapidly initiated into the subject. Definitions are given only as needed.

The selection and arrangement of theorems is such as to meet the general demand of teachers, as expressed through the Mathematical Associations of the country.

Most of the proofs have been given in full. Proofs of some of the easier theorems and constructions are left as exercises for the student, or are given in an incomplete form; but in every case in which the proof is not complete, the incompleteness is specifically stated. The authors believe that the proofs of most of the propositions should be complete, first, in order to serve as models for the handling of exercises; second, to prevent the serious error of making the student feel contented with loose and slipshod reasoning which defeats the main purpose of instruction in geometry; and third, as an excellent means of reviewing the previous theorems on which they depend.

The indirect method of proof is consistently applied. The usual method of proving such propositions as Arts. 189 and

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415, *e.g.*, is confusing to the student. The method used here is convincing and clear.

The exercises are carefully selected. In choosing the exercises, each of the following groups has been given due importance:

(*a*) Concrete exercises, including numerical problems and problems of construction.

(*b*) So-called practical problems, such as indirect measurements of heights and distances by means of equal and similar triangles, drawing to scale as an application of similar figures, problems from physics, from design, etc.

(*c*) The traditional exercises given in a more or less abstract setting.

The arrangement of the exercises is pedagogical. Exercises of a rather easy nature are placed immediately after the theorems of which they are applications, instead of being grouped together without regard to the principles involved in them. In many instances the exercises are so arranged as to constitute a careful line of development, leading gradually from a very simple construction or exercise to others that are more difficult. For the benefit of the brighter pupils, however, and for review classes, large lists of more or less difficult exercises are grouped at the end of each book.

The definitions of plane closed figures are unique. The student's natural conception of a plane closed figure is not the boundary line only, nor the plane only, but the *whole figure* composed of the boundary line and the plane bounded. All definitions of closed figures involve this idea, which is entirely consistent with the higher mathematics.

The numerical treatment of magnitudes is explicit, the fundamental principles being definitely assumed (Art. 336, proof in Appendix, Art. 595). This procedure is novel and is believed to be the only logical, and at the same time teachable, method of dealing with incommensurables. Teachers who find these subjects too difficult, however, can easily omit them without interruption of sequence.

The area of a rectangle is introduced by actually measuring it, thereby obtaining its measure-number. This method permits the same order of theorems and corollaries as is used in the parallelogram and triangle. The *correlation with arithmetic* in this connection is valuable. The number concepts already found so useful and practical in the modern treatment of ratio and proportion have been developed in connection with areas, as well as in other portions of the book

Proofs of the superposition theorems and the concurrent line theorems will be found exceptionally accurate and complete.

The many historical notes are such as will add life and interest to the work.

The carefully arranged summaries throughout the book, and the collection of formulas of plane geometry at the end of the book, it is hoped, will be found helpful to teacher and student alike.

Argument and reasons are arranged in parallel form. This arrangement gives a definite model for proving exercises, renders the careless omission of the reasons in a demonstration impossible, leads to accurate thinking, and greatly lightens the labor of reading papers.

Every construction figure contains all necessary construction lines. This method keeps constantly before the student a model for his construction work, and distinguishes between a figure for a construction and a figure for a theorem.

The mechanical arrangement is such as to give the student every possible aid in comprehending the subject matter.

The grateful acknowledgment of the authors is due to many friends for helpful suggestions; especially to Miss Grace A. Bruce of the Wadleigh High School, New York City; to Mr. Edward B. Parsons of the Boys' High School, Brooklyn; and to Professor McMahon of Cornell University.

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SYMBOLS AND ABBREVIATIONS

$=$	equals, equal to, is equal to.
$\neq$	does not equal.
$>$	greater than, is greater than.
$<$	less than, is less than.
$\Rightarrow$	equivalent, equivalent to, is equivalent to.
$\sim$	similar, similar to, is similar to.
$\underline{\propto}$	is measured by.
$\perp$	perpendicular, perpendicular to, is perpendicular to.
$\perp$s	perpendiculars.
$\parallel$	parallel, parallel to, is parallel to.
$\parallel$s	parallels.
$\ldots$	and so on (sign of continuation).
$\because$	since.
$\therefore$	therefore.
$\frown$	arc; $\overset{\frown}{AB}$, arc AB.
▱, ▱s	parallelogram, parallelograms.
$\odot$, $\odot$s	circle, circles.
$\angle$, $\angle$s	angle, angles.
$\triangle$, $\triangle$s	triangle, triangles.

rt.	right.
str.	straight.
ext.	exterior.
int.	interior.
alt.	alternate.
def.	definition.
ax.	axiom.
post.	postulate.
hyp.	hypothesis.
prop.	proposition.
prob.	problem.
th.	theorem.
cor.	corollary.
cons.	construction.
ex.	exercise.
fig.	figure.
iden.	identity.
comp.	complementary.
sup.	supplementary.
adj.	adjacent.
homol.	homologous.

Q.E.D. Quod erat demonstrandum, *which was to be proved.*

Q.E.F. Quod erat faciendum, *which was to be done.*

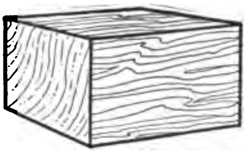
The signs $+$, $-$, $\times$, $\div$ have the same meanings as in algebra.

PLANE GEOMETRY

INTRODUCTION

1. The Subject Matter of Geometry. In geometry, although we shall continue the use of arithmetic and algebra, our main work will be a study of what will later be defined (§ 13) as geometric figures. The student is already familiar with the physical objects about him, such as a ball or a block of wood. By a careful study of the following exercise, he may be led to see the relation of such physical solids to the geometric figures with which he must become familiar.

Exercise. Look at a block of wood (or a chalk box). Has it weight? color? taste? shape? size? These are called *properties* of the solid. What do we call such a solid? *A physical solid.* Can you think of the properties of this solid apart from the block of wood? Imagine the block removed. Can you imagine the space which it occupied? What name would you give to this space? *A geometric solid.* What properties has it that the block possessed? *Shape and size.* What is it that separates this geometric solid from surrounding space? How thick is this surface? How many surfaces has the block? Where do they intersect? How many intersections are there? How wide are the intersections? how long? What is their name? *They are lines.* Do these lines intersect? where? How wide are these intersections? how thick? how long? Can you say *where* this one is and so distinguish from *where* that one is? What is its name? *It is a point.*



If you move the block through space, what will it generate as it moves? What will the surfaces of the block generate? all of them? Can you move a surface so that it will not generate a solid? *Yes, by moving it along itself.* What will the edges of the block generate? Can you move an edge so that it will not generate a surface? What will the corners generate? Can you move a point so that it will not generate a line?

FOUR FUNDAMENTAL GEOMETRIC CONCEPTS

2. The space in which we live, although boundless and unlimited in extent, may be thought of as divided into parts. A physical solid occupies a limited portion of space. The portion of space occupied by a physical solid is called a **geometric solid.**

3. A geometric solid has length, breadth, and thickness. It may also be divided into parts. The boundary of a solid is called a **surface.**

4. A surface is no part of a solid. It has length and breadth, but no thickness. It may also be divided into parts. The boundary of a surface is called a **line.**

5. A line is no part of a surface. It has length only. It may also be divided into parts. The boundary or extremity of a line is called a **point.**

A point is no part of a line. It has neither length, nor breadth, nor thickness. It cannot be divided into parts. It is position only.

THE FOUR CONCEPTS IN REVERSE ORDER

6. As we have considered geometric solid independently of surface, line, and point, so we may consider point independently, and from it build up to the solid.

A small dot made with a sharp pencil on a sheet of paper represents approximately a **geometric point.**

7. If a point is allowed to move in space, the path in which it moves will be a **line.**

A piece of fine wire, or a line drawn on paper with a sharp pencil, represents approximately a geometric line. This, however fine it may be, has *some* thickness and is not therefore an *ideal*, or geometric, line.

8. If a line is allowed to move in space, its path in general will be a **surface**

9. If a surface is allowed to move in space, its path in general will be a **geometric solid.**

10. A solid has threefold extent and so is said to have three dimensions; a surface has twofold extent and is said to have two dimensions; a line has onefold extent or one dimension; a point has no extent and has therefore no dimensions.

11. The following may be used as working definitions of these four fundamental concepts:

A **geometric solid** is a limited portion of space.

A **surface** is that which bounds a solid or separates it from an adjoining solid or from the surrounding space.

A **line** is that which has length only.

A **point** is position only

DEFINITIONS AND ASSUMPTIONS

12. The primary object of elementary geometry is to determine, by a definite process of reasoning that will be introduced and developed later, the properties of geometric figures. In all logical arguments of this kind, just as in a debate, certain fundamental principles are agreed upon at the outset, and upon these as a foundation the argument is built. In elementary geometry these fundamental principles are called *definitions* and *assumptions.*

The assumptions here mentioned are divided into two classes, *axioms* and *postulates.* These, as well as the definitions, will be given throughout the book as occasion for them arises.

13. Def. A **geometric figure** is a point, line, surface, or solid, or a combination of any or all of these.

14. Def. **Geometry** is the science which treats of the properties of geometric figures.

15. Def. A **postulate** may be defined as the assumption of the possibility of performing a certain geometric operation.

Before giving the next definition, it will be necessary to introduce a postulate.

16. Transference postulate. *Any geometric figure may be moved from one position to another without change of size or shape.*

17. Def. Two geometric figures are said to **coincide** if, when either is placed upon the other, each point of one lies upon some point of the other.

18. Def. Two geometric figures are **equal** if they can be made to coincide.

19. Def. The process of placing one figure upon another so that the two shall coincide is called **superposition.**

This is an *imaginary* operation, no actual movement taking place.

LINES

20. A line is usually designated by two capital letters, as line AB. It may be designated also by a small letter placed somewhere on the line, as line a.

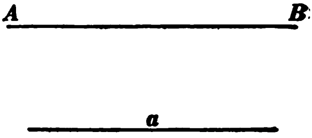


FIG. 1.

21. Straight Lines. In § 7 we learned that a piece of fine wire or a line drawn on a sheet of paper represented approximately a geometric line. So also a geometric *straight line* may be represented approximately by a string stretched taut between two points, or by the line made by placing a *ruler* (also called a *straightedge*) on a flat surface and drawing a sharp pencil along its edge.

22. Questions. How does a gardener test the straightness of the edge of a flower bed? How does he get his plants set out in straight rows? How could you test the straightness of a wire? Can you think of a wire not straight, but of such shape that you could cut out a piece of it and slip it along the wire so that it would always fit? If you reversed this piece, so that its ends changed places, would it still fit along the entire length of the wire? If you turned it over, would it fit? Would the piece cut out fit under these various conditions if the wire were straight?

23. Def. **A straight line is a** line such that, if any portion of it is placed with its ends in the line, the entire portion so placed will lie in the line, however it may be applied.

Thus, if *AB* is a straight line, and if any portion of *AB*, as *CD*, is placed on any other part of *AB*, with its ends in *AB*, every point of *CD* will lie in *AB*.

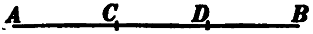


Fig. 2.

A straight line is called also a **right line.** The word line, unqualified, is understood to mean straight line.

24. Straight line postulate. *A straight line may be drawn from any one point to any other.*

25. Draw a straight line *AB*. Can you draw a second straight line from *A* to *B*? If so, where will every point of the second line lie (§ 23)? It then follows that:

Only one straight line can be drawn between two points; i.e. *a straight line is* **determined** *by two points.*

26. Draw two straight lines *AB* and *CD* intersecting in point *P*. Show that *AB* and *CD* cannot have a second point in common (§ 23). It then follows that:

Two intersecting straight lines can have only one point in common; i.e. *two intersecting straight lines* **determine** *a point.*

27. Def. A limited portion of a straight line is called **a line segment**, or simply **a line**, or a **segment.** Thus, in Fig. 2, *AC*, *CD*, and *DB* are line segments.

28. Def. Two line segments which lie in the same straight line are said to be **collinear segments.**

29. Def. A **curved line** (or curve) is a line no portion of which is straight, as *GH*.

30. Def. A **broken line** is a line made up of different successive straight lines, as *KL*.

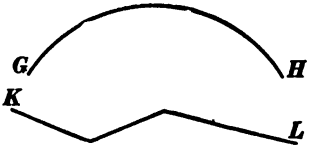


Fig. 3.

31. Use of Instruments. Only two instruments are permitted in the constructions of plane geometry: the *ruler* or *straightedge* for drawing a straight line, already spoken of in § 21; and the *compasses*, for constructing circles or arcs of circles, and for transferring line segments from one position to another.

Thus to add two lines, as AB and CD, draw, with a ruler, a straight line OX. Place one leg of the compasses at A and the other at B. Next place one leg at O and cut off segment OM equal to AB. In a similar manner lay off MN equal to CD. Then

$AB+CD=OM+MN=ON$.

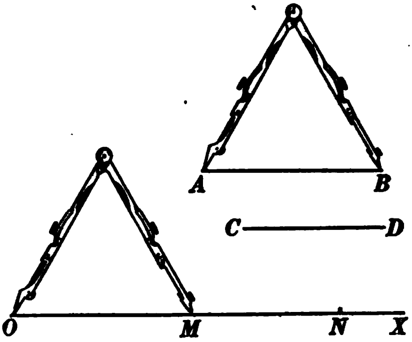


FIG. 4.

Show how to subtract AB from ON. What is the remainder?

Ex. 1. Can a straight line move so that its path will not be a surface? If so, how?

Ex. 2. Can a curved line move in space so that its path will not be a surface? If so, how?

Ex. 3. Can a broken line move in space so that its path will not be a surface?

Ex. 4. Draw three lines as AB, CD, and EF. Construct the sum of AB and CD; of AB and EF; of AB, CD, and EF.

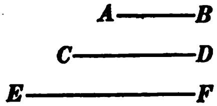


FIG. 5.

Ex. 5. Construct: (*a*) the difference between AB and CD; (*b*) the difference between CD and EF; (*c*) the difference between AB and EF. Add the results obtained for (*a*) and (*b*) and see whether the sum is the result obtained for (*c*).

Ex. 6 Draw a line twice as long as AB (the sum of AB and AB); three times as long as AB.

SURFACES

32. Plane Surface. It is well known that the carpenter's straightedge is applied to surfaces to test whether they are flat and even. If, no matter where the straightedge is placed on the surface, it always fits, the surface is called a plane. Now if we should use a powerful magnifier, we should doubtless discover that in certain places the straightedge did not exactly fit the surface on which it was placed. A sheet of fine plate glass more nearly approaches the ideal.

33. Questions. Test the surface of the blackboard with a ruler to see whether it is a plane. How many times must you apply the ruler? Can you think of a surface such that the ruler would fit in some positions (a great many) but not in all? Can you think of a surface not plane but such that a piece of it could be cut out and slipped along the rest so that it would fit? Would it fit if turned over (inside out)?

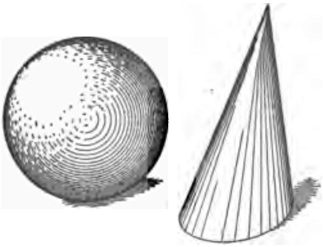


FIG. 6.

34. Def. A **plane surface** (or plane) is a surface of unlimited extent such that whatever two of its points are taken, a straight line joining them will lie wholly in the surface.

35. Def. A **curved surface** is a surface no portion of which is plane.

36. Def. A **plane figure** is a geometric figure all of whose points lie in one plane. **Plane Geometry** treats of plane figures.

37. Def. A **rectilinear figure** is a plane figure all the lines of which are straight lines.

Ex. 7. How can a plane move in space so that its path will not be a solid?

Ex. 8. Can a curved surface move in space so that its path will not be a solid? If so, how?

ANGLES

38. Def. An **angle** is the figure formed by two straight lines which diverge from a point.

The point is the **vertex** of the angle and the lines are its **sides.**

39. An angle may be **designated** by a number placed within it, as angle 1 and angle 2 in Fig. 7, and angle 3 in Fig. 8. Or three letters may be used, one on each side and one at the vertex, the last being read between the other two; thus in Fig. 7, angle 1 may be read angle *ABC*, and angle 2, angle *CBD*. An angle is often designated also by the single letter at its vertex, when no other angle has the same vertex, as angle *F* in Fig. 8.

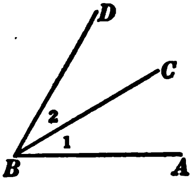


FIG. 7.

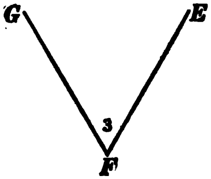


FIG. 8.

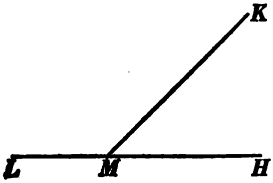


FIG. 9.

40. Revolution postulate. *A straight line may revolve in a plane, about a point as a pivot, and when it does revolve continuously from one position to another, it passes once and only once through every intermediate position.*

41. A clear notion of the *magnitude* of an angle may be obtained by imagining that its two sides were at first collinear, and that one of them has *revolved* about a point common to the two. Thus in Fig. 8. we may imagine *FG* first to have been in the position *FE* and then to have revolved about *F* as a pivot to the position *FG*.

42. Def. Two angles are **adjacent** if they have a common vertex and a common side which lies between them; thus in Fig. 7, angle 1 and angle 2 are adjacent; also in Fig. 9, angle *HMK* and angle *KML* are adjacent.

43. Two angles are **added** by placing them so that they are adjacent. Their **sum** is the angle formed by the two sides that are not common; thus in Fig. 10, the sum of angle 1 and angle 2 is angle *ABC*.

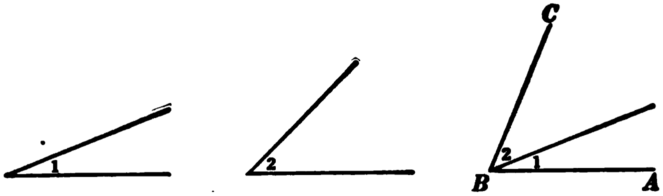


FIG. 10.

44. The **difference** between two angles is found by placing them so that they have a vertex and a side in common but with the common side *not* between the other two. If the other two sides then happen to be collinear, the difference between the angles is zero and the angles are **equal**. If the other sides are not collinear, the angle which they form is the **difference** between the two angles compared; thus in Fig. 11, the difference between angle 1 and angle 2 is angle *ABC*.

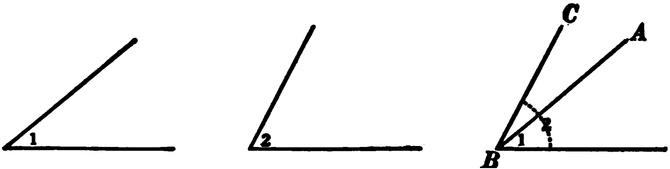


FIG. 11.

45. Def. If one straight line meets another so as to make two adjacent angles equal, each of these angles is a **right angle**, and the lines are said to be **perpendicular** to each other. Thus, if *DC* meets *AB* so that angle *BCD* and angle *DCA* are equal angles, each is a right angle, and lines *AB* and *CD* are said to be perpendicular to each other.

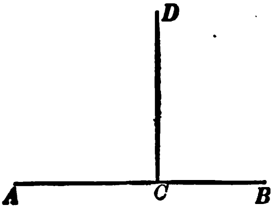


FIG. 12.

46. Def. If two lines meet, but are not perpendicular to each other, they are said to be **oblique** to each other.

47. Def. An **acute angle** is an angle that is less than a right angle; as angle 1, Fig. 13.

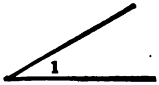


FIG. 13.

48. Def. An **obtuse angle** is an angle that is greater than a right angle and less than two right angles; as angle 2, Fig. 14.

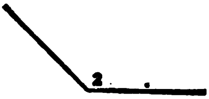


FIG. 14.

49. Def. A **reflex angle** is an angle that is greater than two right angles and less than four right angles; as angle 2, Fig. 15.

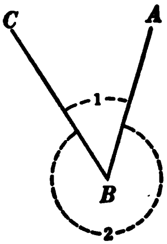


FIG. 15.

50. Note. Two lines diverging from the same point, as BA and BC, Fig. 15, always form two positive angles, as the acute angle 1 and the reflex angle 2. Angle 1 may be thought of as formed by the revolution of a line counter-clockwise from the position BA to the position BC, and should be read angle ABC. Angle 2 may be thought of as formed by the revolution of a line counter-clockwise from the position BC to the position BA, and should be read angle CBA.

51. Def. Acute, obtuse, and reflex angles are sometimes called **oblique angles.**

Ex. 9. (*a*) In Fig. 16, if angle 1 equals angle 2, what kind of angles are they?

(*b*) Make a statement with regard to the lines AB and CD.

(*c*) If angle 3 does not equal angle 4, what kind of angles are they?

(*d*) Make a statement with regard to the lines AB and CE.

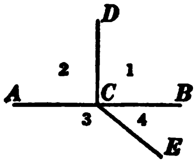


FIG. 16.

Ex. 10. A plumb line is suspended from the top of the blackboard. What kind of angles does it make with a horizontal line drawn on the blackboard? with a line on the blackboard neither horizontal nor vertical?

Ex. 11. Suppose the minute hand of a clock is at twelve. Where may the hour hand be so that the two hands make with each other: (*a*) an acute angle? (*b*) a right angle? (*c*) an obtuse angle?

Ex. 12. Draw: (*a*) a pair of adjacent angles; (*b*) a pair of non-adjacent angles.

Ex. 13. Draw two adjacent angles such that: (*a*) each is an acute angle; (*b*) each is a right angle; (*c*) each is an obtuse angle; (*d*) one is acute and the other right; (*e*) one is acute and the other obtuse.

Ex. 14. In Fig. 17, angle $1 +$ angle $2 = ?$ angle $3 +$ angle $4 = ?$ angle $BAD +$ angle $DAF = ?$ angle $DAF -$ angle $3 = ?$ angle $2 +$ angle $DAF -$ angle $4 = ?$ angle $4 +$ angle $BAE -$ angle $1 = ?$

Ex. 15. Name six pairs of adjacent angles in Fig. 17.

Ex. 16. Draw two non-adjacent angles that have: (*a*) a common vertex; (*b*) a common side; (*c*) a common vertex and a common side.

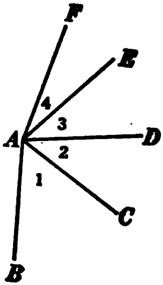


FIG. 17.

52. Def. A line is said to be **bisected** if it is divided into two equal parts.

53. Def. The **bisector of an angle** is the line which divides the angle into two equal angles.*

Ex. 17. Draw a line AB, neither horizontal nor vertical. (*a*) Draw freehand a line perpendicular to AB and not bisecting it: (*b*) a line bisecting AB and not perpendicular to it.

ASSUMPTIONS

54. 1. *Things equal to the same thing, or to equal things, are equal to each other.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are added to unequals, the sums are unequal in the same order.*

5. *If equals are subtracted from unequals, the remainders are unequal in the same order.*

---

* The proof that every angle has but one bisector will be found in the Appendix. § 599.

6. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

7. (*a*) *If equals are multiplied by equals, the products are equal;* (*b*) *if unequals are multiplied by equals, the products are unequal in the same order.*

8. (*a*) *If equals are divided by equals, the quotients are equal;* (*b*) *if unequals are divided by equals, the quotients are unequal in the same order.*

9. *If unequals are added to unequals, the less to the less and the greater to the greater, the sums are unequal in the same order.*

10. *If three magnitudes of the same kind are so related that the first is greater than the second and the second greater than the third, then the first is greater than the third.*

11. *The whole is equal to the sum of all its parts.*

12. *The whole is greater than any of its parts.*

13. *Like powers of equal numbers are equal, and like roots of equal numbers are equal.*

14. **Transference postulate.** *Any geometric figure may be moved from one position to another without change of size or shape.* (See § 16.)

15. **Straight line postulate I.** *A straight line may be drawn from any one point to any other.* (See § 24.)

16. **Straight line postulate II.** *A line segment may be prolonged indefinitely at either end.*

17. **Revolution postulate.** *A straight line may revolve in a plane, about a point as a pivot, and when it does revolve continuously from one position to another, it passes once and only once through every intermediate position.* (See § 40.)

55. Assumptions 1–13 are usually called axioms. That is, an **axiom** may be defined as a statement whose truth is assumed.*

Ex. 18. Illustrate the first five assumptions above by using arithmetical numbers only.

Ex. 19. Illustrate the next five by using general numbers (letters) only.

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* See Appendix, § 600.

DEMONSTRATIONS

56. It has been stated (§ 12) that the fundamental principles agreed upon at the outset as forming the basis of the logical arguments in geometry are called definitions, axioms, and postulates. Every new proposition advanced, whether it is a statement of a truth or a statement of something to be performed, must by a process of reasoning be shown to depend upon these fundamental principles. This process of reasoning is called a **proof** or **demonstration.** After the truth of a statement has thus been established, it in turn may be used to establish new truths.

The propositions here mentioned are divided into two classes, *theorems* and *problems.*

57. Def. A **theorem** is a statement whose truth is required to be proved or demonstrated. For example, "*If two angles of a triangle are equal, the sides opposite are equal*" is a theorem.

There are two parts to every theorem: the **hypothesis,** or the conditional part; and the **conclusion,** or the part to be proved. In the theorem just quoted, "*If two angles of a triangle are equal*" is the hypothesis; and "*the sides opposite are equal*" is the conclusion.

Ex. 20. Write out carefully the hypothesis and the conclusion of each of the following:

(*a*) If you do your duty at all times, you will be rewarded.
(*b*) If you try to memorize your proofs, you will never learn geometry.
(*c*) You must suffer if you disobey a law of nature.
(*d*) Things equal to the same thing are equal to each other.
(*e*) All right angles are equal.

---

58. Def. A **corollary** is a statement of a truth easily deduced from another truth. Its correctness, like that of a theorem, must be proved.

59. Def. A **problem,** in general, is a question to be solved. As applied to geometry, problems are of two kinds, namely, problems of construction and problems of computation.

60. From the revolution postulate (§ 40) and from the definition of a perpendicular (§ 45) we may deduce the following corollaries:

61. Cor. I. *At every point in a straight line there exists a perpendicular to the line.*

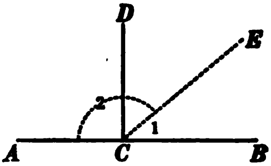

FIG. 18.

Given line AB and point C in AB.

To prove that there exists a $\perp$ to AB at C, as CD.

Let CE (Fig. 18) meet AB at C, so that $\angle 1 < \angle 2$. Then if CE is revolved about C as a pivot toward position CA, $\angle 1$ will continuously increase and $\angle 2$ will continuously decrease (§ 40). $\therefore$ there must be one position of CE, as CD, in which the two $\angle$s formed with AB are equal. In this position $CD \perp AB$ (§ 45).

62. Cor. II. *At every point in a straight line there exists only one perpendicular to the line.*

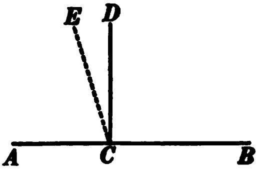

FIG. 19.

Given $CD \perp AB$ at C, *i.e.* $\angle BCD = \angle DCA$.

To prove CD the only $\perp$ to AB at C.

Let CE (Fig. 19) be any line from C other than CD. Let CE fall between CD and CA.* Then $\angle BCE > \angle BCD$; *i.e.* $> \angle DCA$. And $\angle ECA < \angle DCA$ (§ 54, 12). $\therefore$ $\angle$s BCE and ECA are not equal and CE is not $\perp$ to AB (§ 45).

63. §§ 61 and 62 may be combined in one statement as follows:

At every point in a straight line there exists one and only one perpendicular to the line.

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* A similar proof may be given if we suppose CE to fall between CB and CD.

64. Cor. III. *All right angles are equal.*

FIG. 20.

Given $\angle AOB$ and $\angle CPD$, any two rt. $\angle s$.
To prove $\angle AOB = \angle CPD$.

Place $\angle CPD$ upon $\angle AOB$ so that point P shall fall upon point O, and so that PC shall be collinear with OA. Then PD and OB, both being $\perp$ to OA at O, must be collinear (§ 62). $\therefore$ the two $\angle s$ coincide and are equal (§ 18).

65. Cor. IV. *If one straight line meets another straight line, the sum of the two adjacent angles is two right angles.*

Given str. line CD meeting str. line AB at C, forming $\angle s$ BCD and DCA.
To prove $\angle BCD + \angle DCA = 2$ rt. $\angle s$.

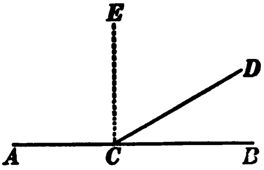


FIG. 21.

Let CE be $\perp$ to AB at C (§ 63).

$$\text{Then } \angle BCD + \angle DCE = 1 \text{ rt. } \angle;$$
$$\therefore \angle BCD = 1 \text{ rt. } \angle - \angle DCE.$$
$$\text{Again } \angle DCA = 1 \text{ rt. } \angle + \angle DCE.$$
$$\therefore \angle BCD + \angle DCA = 2 \text{ rt. } \angle s.$$

66. Cor. V. *The sum of all the angles about a point on one side of a straight line passing through that point equals two right angles.*

Given $\angle s$ 1, 2, 3, 4, 5, and 6, all the $\angle s$ about point O on one side of str. line CA.
To prove $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 + \angle 6 = 2$ rt. $\angle s$.

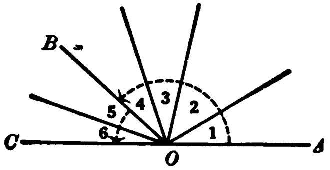


FIG. 22.

The sum of the six $\angle s$ (Fig.22) is equal to $\angle AOB + \angle BOC$.

67. Cor. VI. *The sum of all the angles about a point equals four right angles.*

Prolong one of the lines through the vertex and apply Cor. V.

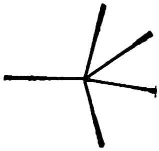


FIG. 23.

68. Def. Two angles whose sum is one right angle are called **complementary** angles, as angles 1 and 2, Fig. 25. Either of two such angles is said to be the **complement** of the other.

69. Def. Two angles whose sum is two right angles are called **supplementary** angles, as angles 1 and 2, Fig. 26. Either of two such angles is said to be the **supplement** of the other.

An angle that is equal to the sum of two right angles is sometimes called a **straight angle**, as angle *ABC*, Fig. 26.

70. Def. Two angles are said to be **vertical** if the sides of each are the prolongations of the sides of the other; thus, in Fig. 24, angles 1 and 2 are vertical angles, and angles 3 and 4 are likewise vertical angles.

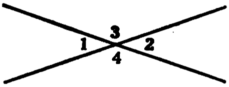


FIG. 24.

71. Def. An **angle of one degree** is one ninetieth of a right angle. A right angle, therefore, contains 90 angle degrees.

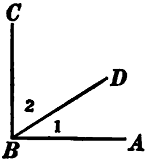


FIG. 25.

Ex. 21. In Fig. 25, angle *ABC* is a right angle. If angle 1 = 40°, how many degrees are there in angle 2? How many degrees, then, are there in the complement of an angle of 40°?

Ex. 22. How many degrees are there in the complement of 20°? of 35°? of $a°$? of $\frac{4}{5}$ right angles? of k right angles?

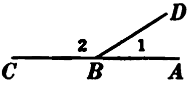


FIG. 26.

Ex. 23. In Fig. 26, angle 1 + angle 2, or angle *ABC*, equals 2 right angles. If angle 1 = 40°, how many degrees are there in angle 2? How many degrees, then, are there in the supplement of an angle of 40°?

Ex. 24. How many degrees are there in the supplement of 20°? of 140°? of $a°$? of n right angles? Give these answers in right angles.

Ex. 25. In the accompanying diagram:

(*a*) If angle $1 = 65°$, how many degrees are there in each of the other three angles?

(*b*) If angle $2 = m°$, how many right angles are there in each of the other three angles?

(*c*) If angle $3 = n$ right angles, how many degrees are there in each of the other three angles?

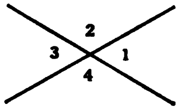


Fig. 27.

Ex. 26. Compare the supplement of an angle of 50° with its complement. Draw a diagram showing the complement and the supplement of an acute angle, ABC.

Ex. 27. Criticize the following exercise: How many degrees are there in an angle whose complement is $\frac{3}{2}$ of its supplement? in an angle whose supplement is $\frac{2}{3}$ of its complement?

Ex. 28. If one straight line meets another straight line so that one angle is double its adjacent angle, how many degrees are there in each of the two adjacent angles?

Ex. 29. How many degrees are there in an angle whose complement and supplement together equal 194°?

Ex. 30. How many degrees are there in an angle which is $\frac{5}{4}$ of its complement?

Ex. 31. How many degrees are there in an angle whose supplement is nineteen times its complement?

Ex. 32. If there are only five angles about a point, and each differs by 15° from an angle adjacent, how many degrees are there in each angle?

Ex. 33. If there are only six angles about a point and they are all equal, how large is each angle?

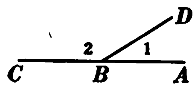


Fig. 28.

72. Def. Angles that are supplementary and adjacent are called **supplementary-adjacent** angles, as $\angle$s 1 and 2.

Ex. 34. If two angles are supplementary-adjacent and one of them is one half a right angle, how large is the other?

Ex. 35. Suppose that only three angles are formed about a point on one side of a straight line passing through the point. The greatest is four times the least, and the remaining one is 12° more than the least. How many degrees are there in each?

Ex. 36. (*a*) Can two angles be supplementary and not adjacent? Illustrate.

(*b*) Can two angles be adjacent and not supplementary? Illustrate.

(*c*) Can two angles be both supplementary and adjacent? Illustrate.

(*d*) Can two angles be neither supplementary nor adjacent? Illustrate.

(*e*) Can two angles be both complementary and adjacent? Illustrate.

(*f*) Can two angles be both complementary and supplementary? Illustrate.

---

73. From the definitions of complementary and supplementary angles may be deduced three additional corollaries as follows:

74. Cor. I. *Complements of the same angle or of equal angles are equal.*

75. Cor. II. *Supplements of the same angle or of equal angles are equal.*

76. Cor. III. *If two adjacent angles are supplementary, their exterior sides are collinear.*

Given $\angle ABC + \angle CBD = 2$ rt. $\angle$s.

To prove BD the prolongation of AB.

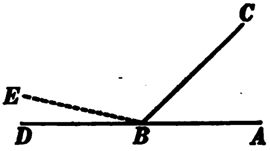


FIG. 29.

If BD is not the prolongation of AB, then some other line from B, as BE, must be its prolongation.

Then $\angle ABC + \angle CBD = 2$ rt. $\angle$s (By hyp.).

Also $\angle ABC + \angle CBE = 2$ rt. $\angle$s (§ 65).

$\therefore \angle CBD = \angle CBE$ (§ 75).

This is impossible (§ 54, 12); *i.e.* the supposition that BE is the prolongation of AB leads, by correct reasoning, to the impossible conclusion that $\angle CBD = \angle CBE$. Hence this supposition itself is false.

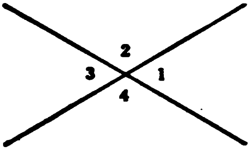
$\therefore$ BD is the prolongation of AB.

BOOK I

RECTILINEAR FIGURES

PROPOSITION I. THEOREM

77. *If two straight lines intersect, the vertical angles are equal.*



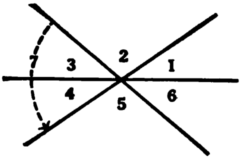
Given two intersecting str. lines, forming the vertical $\angle$s, 1 and 3, also 2 and 4.

To prove $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.

ARGUMENT	REASONS
1. $\angle 1 + \angle 2 = 2$ rt. $\angle$s.	1. If one str. line meets another str. line, the sum of the two adj. $\angle$s is 2 rt. $\angle$s. § 65.
2. $\angle 3 + \angle 2 = 2$ rt. $\angle$s.	2. Same reason as 1.
3. $\therefore \angle 1 + \angle 2 = \angle 3 + \angle 2$.	3. Things equal to the same thing are equal to each other. § 54, 1.
4. $\therefore \angle 1 = \angle 3$.	4. If equals are subtracted from equals, the remainders are equal. § 54, 3.
5. Likewise $\angle 2 = \angle 4$. Q.E.D.	5. By steps similar to 1, 2, 3, and 4.

78. Cor. *If two straight lines intersect so that any two adjacent angles thus formed are equal, all the angles are equal, and each angle is a right angle.*

Ex. 37. If three straight lines intersect at a common point, the sum of any three angles, no two of which are adjacent, as 1, 3, and 5, in the accompanying diagram, is equal to two right angles.



Ex. 38. In the same diagram, show that:

(*a*) Angle 1 + angle 3 = angle 7.

(*b*) Angle 7 – angle 4 = angle 6.

(*c*) Angle 7 + angle 4 – angle 3 = twice angle 1.

(*d*) Angle 2 + angle 4 + angle 6 = 2 right angles.

Ex. 39. The line which bisects one of two vertical angles bisects the other also.

Ex. 40. The bisectors of two adjacent complementary angles form an angle of 45°.

Ex. 41. Show the relation of the bisectors of two adjacent supplementary angles to each other.

Ex. 42. The bisectors of two pairs of vertical angles are perpendicular to each other.

79. The student will observe from Prop. I that the complete solution of a theorem consists of four distinct steps:

(1) *The statement of the theorem,* including a general statement of the hypothesis and conclusion.

(2) *The drawing of a figure* (as *general* as possible), to satisfy the conditions set forth in the hypothesis.

(3) *The application, i.e.* the restatement of the hypothesis and conclusion as applied to the particular figure just drawn, under the headings of "Given" and "To prove."

(4) *The proof,* or demonstration, the argument by which the truth or falsity of a statement is established, with the reason for each step in the argument.

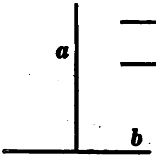
80. The student should be careful to quote the reason for every step in the argument which he makes in proving a theorem. The only reasons admissible are of two kinds:

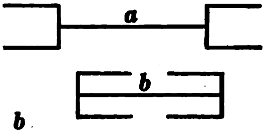
1. Something assumed, *i.e.* definitions, axioms, postulates, and the hypothesis.

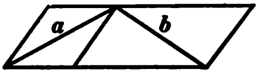
2. Something previously proved, *i.e.* theorems, corollaries, and problems.

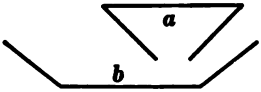
81. The necessity for proof. Some theorems seem evident by merely looking at the figure, and the student will doubtless think a proof unnecessary. The eye, however, cannot always detect error, and *reasoning* enables us to be sure of our conclusions. The danger of trusting the eye is illustrated in the following exercises.*

Ex. 43. In the diagrams given below, tell which line of each pair is the longer, *a* or *b*, and verify your answer by careful measurement.

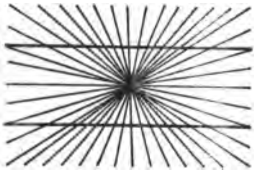


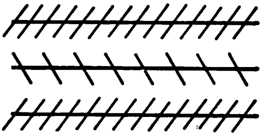




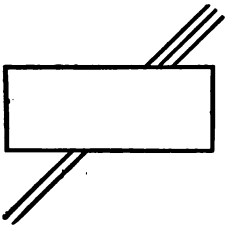


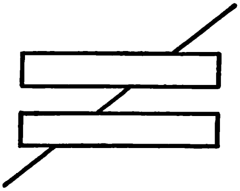
Ex. 44. In the figures below, are the lines everywhere the same distance apart? Verify your answer by using a ruler or a slip of paper.





Ex. 45. In the figures below, tell which lines are prolongations of other lines. Verify your answers.





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* These diagrams are taken by permission from the Report of the Committee on Geometry, Proceedings of the Eighth Meeting of the Central Association of Science and Mathematics Teachers.

POLYGONS. TRIANGLES

82. Def. A line on a plane is said to be **closed if it separates a finite portion of the plane from the remaining portion.**

83. Def. A **plane closed figure** is a plane figure composed of a closed line and the finite portion of the plane bounded by it.

84. Def. A **polygon** is a plane closed figure whose boundary is composed of straight lines only.

The points of intersection of the lines are the **vertices** of the polygon, and the segments of the boundary lines included between adjacent vertices are the **sides** of the polygon.

85. Def. The sum of the sides of a polygon is its **perimeter**.

86. Def. Any angle formed by two consecutive sides and found on the right in passing clockwise around the perimeter of a polygon is called an **interior angle** of the polygon, or, for brevity, an **angle of the polygon.** In Fig. 1, ∠s *ABC*, *BCD*, *CDE*, *DEA*, and *EAB* are interior angles of the polygon.

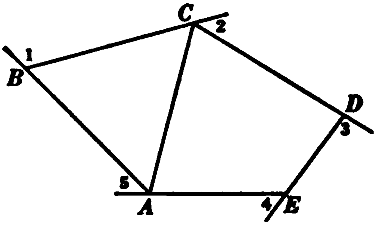


FIG. 1.

87. Def. If any side of a polygon is prolonged through a vertex, the angle formed by the prolongation and the adjacent side is called an **exterior angle** of the polygon.

In Fig. 1, ∠s 1, 2, 3, 4, and 5 are exterior angles.

88. Def. A line joining any two non-adjacent vertices of a polygon is called a **diagonal**; as *AC*, Fig. 1.

89. Def. A polygon which has all of its sides equal is an **equilateral polygon.**

90. Def. A polygon which has all of its angles equal is an **equiangular polygon.**

91. Def. A polygon which is both equilateral and equiangular is a **regular polygon.**

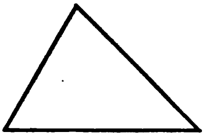
92. Def. A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral**; one of five sides, a **pentagon**; one of six sides, a **hexagon**; and so on.

TRIANGLES CLASSIFIED WITH RESPECT TO SIDES

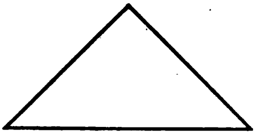
93. Def. A triangle having no two sides equal is a **scalene triangle.**

94. Def. A triangle having two sides equal is an **isosceles triangle.** The equal sides are spoken of as **the sides** * of the triangle. The angle between the equal sides is the **vertex** angle, and the side opposite the vertex angle is called the **base.**

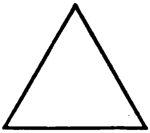
95. Def. A triangle having its three sides equal is an **equilateral triangle.**



Scalene



Isosceles



Equilateral

TRIANGLES CLASSIFIED WITH RESPECT TO ANGLES

96. Def. A **right triangle** is a triangle which has a right angle. The side opposite the right angle is called the **hypotenuse.** The other two sides are spoken of as **the sides** † of the triangle.

97. Def. An **obtuse triangle** is a triangle which has an obtuse angle.

98. Def. An **acute triangle** is a triangle in which *all* the angles are acute.

---

* The equal sides are sometimes called the *arms* of the isosceles triangle. This term will be used occasionally in the exercises.

† Sometimes the sides of a right triangle including the right angle are called the *arms* of the triangle. This term will be found in the exercises.

Ex. 46. Draw a scalene triangle freehand: (*a*) with all its angles acute and with its shortest side horizontal; (*b*) with one right angle.

Ex. 47. Draw an isosceles triangle: (*a*) with one of its arms horizontal and one of its angles a right angle; (*b*) with one angle obtuse.

---

99. Def. The side upon which a polygon is supposed to stand is usually called its **base**; however, since a polygon may be supposed to stand upon any one of its sides, any side may be considered as its base.

The angle opposite the base of a triangle is the **vertex angle**, and the vertex of the angle is called the **vertex** of the triangle.

100. Def. The **altitude** of a triangle is the perpendicular to its base from the opposite vertex. In general any side of a triangle may be considered as its base. Thus in triangle *EFG*, if *FG* is taken as base, *EH* is the altitude; if *GE* is taken as base, *FK* will be the altitude; if *EF* is taken as base, the third altitude can be drawn. Thus every triangle has *three* altitudes.

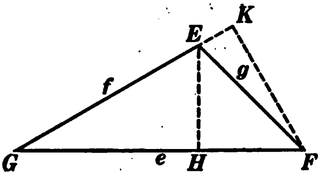


FIG. 1.

It will be proved later that one perpendicular, and only one, can be drawn from a point to a line.

101. The sides of a triangle are often designated by the small letters corresponding to the capitals at the opposite vertices; as, sides *e*, *f*, and *g*, Fig. 1.

Ex. 48. Draw an acute triangle; draw its three altitudes freehand. Do they seem to meet in a point? Where is this point located?

Ex. 49. Draw an obtuse triangle; draw its three altitudes freehand. Do they meet in a point? Where is this point located?

Ex. 50. Where do the three altitudes of a right triangle meet?

---

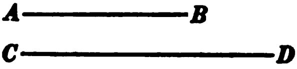
102. Def. The **medians** of a triangle are the lines from the vertices of the triangle to the mid-points of the opposite sides.

103. Superposition. When certain parts of two figures are given equal, we can determine by a process of pure *reason* whether the two figures may be made to coincide.

This process is *far more accurate* than the actual transference of figures, for we are free from physical errors such as have been referred to in § 81.

Problem. Given line AB less than CD. Apply AB to CD.

Solution. Place point A upon point C. Make AB collinear with CD and let B fall toward D. Then B will fall between C and D, *because* $AB < CD$.

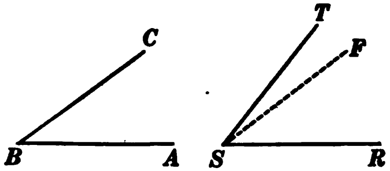


Question. Under what hypothesis would B fall on D? beyond D?

Problem. Given angle ABC less than angle RST. Apply angle ABC to angle RST.

Solution. Place point B upon point S and make BA collinear with SR.

Then BC will fall *between* SR and ST, *because* $\angle ABC < \angle RST$.



Ex. 51. Solve the problem above by first making BC collinear with ST. Under what hypothesis would BA fall on SR? outside of angle RST? within angle RST? Illustrate each answer by a diagram. Can you *choose* where BA will fall after you have put BC on ST? Could you have chosen where BA should fall at first?

104. Note. In applying one figure to another, always begin with a *line*. Place one end of it on one end of another line and make the two lines collinear on the same side of the point.

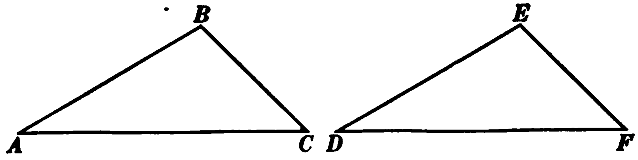
Throughout the process, *determine first* the *direction* each line will take, and then where its *end* will fall.

If two lines are given equal, one may be placed upon the other, end on end, for the lines will coincide.

Ex. 52. Given two triangles ABC and DEF, such that: (1) $AC = DF$, angle A is greater than angle D, angle C is less than angle F; (2) $AC = DF$, angle $A =$ angle D, angle C is less than angle F; (3) $AC = DF$, angle $A =$ angle D, angle $C =$ angle F. Apply triangle ABC to triangle DEF and draw a diagram to illustrate each case.

Proposition II. Theorem

105. *Two triangles are equal if two angles and the included side of one are equal respectively to two angles and the included side of the other.*



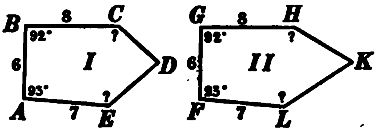
Given $\triangle$s ABC and DEF, $AC = DF$, $\angle A = \angle D$, and $\angle C = \angle F$.

To prove $\triangle ABC = \triangle DEF$.

Argument	Reasons
1. Place $\triangle ABC$ upon $\triangle DEF$ so that AC shall fall upon its equal DF, A upon D, C upon F.	1. Any geometric figure may be moved from one position to another without change of size or shape. § 54, 14.
2 Then AB will become collinear with DE, and B will fall somewhere on DE, or on its prolongation.	2. $\angle A = \angle D$, by hyp.
3. Also CB will become collinear with FE, and B will fall somewhere on FE, or on its prolongation.	3. $\angle C = \angle F$, by hyp.
4. $\therefore$ point B must fall on point E.	4. Two intersecting str. lines determine a point. § 26.
5. $\therefore \triangle ABC = \triangle DEF$.	5. Two geometric figures are equal if they can be made to coincide. § 18.
Q.E.D.	

Ex. 53. In the figure for Prop. II, assume $AB = DE$, angle $A =$ angle D, and angle $B =$ angle E; repeat the proof by superposition, marking the lines with colored crayon as soon as their positions are determined.

Ex. 54. Place polygon I upon polygon II so that some part of I shall fall upon its equal in II. Discuss the resulting positions of the remaining parts of the figure.



Ex. 55. If two quadrilaterals have three sides and the included angles of one equal respectively to three sides and the included angles of the other, and arranged in the same order, are the quadrilaterals equal? Prove.

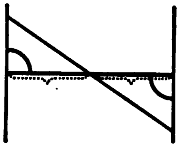
---

106. Note. The method of superposition should be used for proving fundamental propositions only. In proving other propositions it is necessary to show merely that certain conditions are present and to quote theorems, previously proved, which state conclusions regarding such conditions.

Ex. 56. If at any point in the bisector of an angle a perpendicular to the bisector is drawn meeting the sides of the angle, the two triangles thus formed will be equal.

Ex. 57. If equal segments, measured from the point of intersection of two lines, are laid off on one of the lines, and if perpendiculars to this line are drawn at the ends of these segments, two equal triangles will be formed.

Ex. 58. If at the ends of a straight line perpendiculars to it are drawn, these perpendiculars will cut off equal segments upon any line which bisects the given line and is not perpendicular to it.



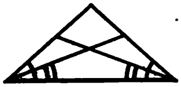


FIG. 1.

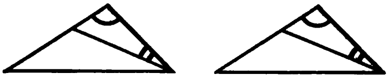


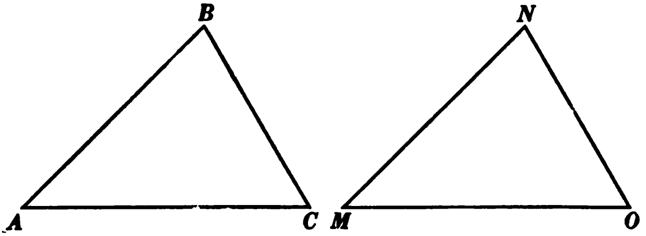
FIG. 2.

Ex. 59. If two angles of a triangle are equal, the bisectors of these angles are equal (Fig. 1).

Ex. 60. If two triangles are equal, the bisector of any angle of one is equal to the bisector of the corresponding angle of the other (Fig. 2).

PROPOSITION III. THEOREM

107. *Two triangles are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given $\triangle s$ ABC and MNO, $AB = MN$, $AC = MO$, and $\angle A = \angle M$.

To prove $\triangle ABC = \triangle MNO$.

ARGUMENT	REASONS
1. Place $\triangle ABC$ upon $\triangle MNO$ so that AC shall fall upon its equal MO, A upon M, C upon O.	1. Any geometric figure may be moved from one position to another without change of size or shape. § 54, 14.
2. Then AB will become collinear with MN.	2. $\angle A = \angle M$, by hyp.
3. Point B will fall on point N.	3. $AB = MN$, by hyp.
4. $\therefore BC$ will coincide with NO.	4. Only one str. line can be drawn between two points. § 25.
5. $\therefore \triangle ABC = \triangle MNO$. Q.E.D.	5. Two geometric figures are equal if they can be made to coincide. § 18.

108. Cor. *Two right triangles are equal if the two sides including the right angle of one are equal respectively to the two sides including the right angle of the other.*

Ex. 61. Prove Prop. III by placing AB upon MN.

109. Def. In equal figures, the points, lines, and angles in one which, when superposed, coincide respectively with points, lines, and angles in the other, are called **homologous parts.** Hence:

110. *Homologous parts of equal figures are equal.*

Ex. 62. If two straight lines bisect each other, the lines joining their extremities are equal in pairs.

HINT. To prove two lines or two angles equal, try to find two triangles, each containing one of the lines or one of the angles. If the triangles can be proved equal, and the two lines or two angles are *homologous* parts of the triangles, then the lines or angles are equal. The parts given equal may be more easily remembered by marking them with the same symbol, or with colored crayon.

Ex. 63. In case the lines in Ex. 62 are perpendicular to each other, what additional statement can you make? Prove its correctness.

Ex. 64. If equal segments measured from the vertex are laid off on the sides of an angle, and if their extremities are joined to any point in the bisector of the angle, two equal triangles will be formed.

Ex. 65. If two medians of a triangle are perpendicular to the sides to which they are drawn, the triangle is equilateral.

Ex. 66. If equal segments measured from the vertex are laid off on the arms of an isosceles triangle, the lines joining the ends of these segments to the opposite ends of the base will be equal (Fig. 1).

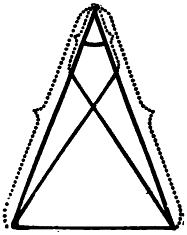


FIG. 1.

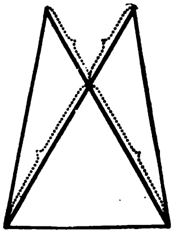
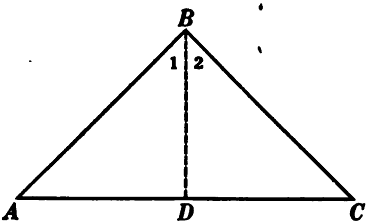


FIG. 2.

Ex 67. Extend Ex. 66 to the case in which the equal segments are laid off on the arms prolonged through the vertex (Fig. 2).

PROPOSITION IV. THEOREM

111. *The base angles of an isosceles triangle are equal*



Given isosceles $\triangle ABC$, with AB and BC its equal sides.
To prove $\angle A = \angle C$.

ARGUMENT	REASONS
1. Let BD bisect $\angle ABC$.	1. Every $\angle$ has but one bisector. § 53.
2. In $\triangle$s ABD and DBC, $AB = BC$.	2. By hyp.
3. $BD = BD$.	3. By iden.
4. $\angle 1 = \angle 2$.	4. By cons.
5. $\therefore \triangle ABD = \triangle DBC$.	5. Two $\triangle$s are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other. § 107.
6. $\therefore \angle A = \angle C$. Q.E.D.	6. Homol. parts of equal figures are equal. § 110.

112. **Cor. I.** *The bisector of the angle at the vertex of an isosceles triangle is perpendicular to the base and bisects it.*

113. **Cor. II.** *An equilateral triangle is also equiangular.*

Ex. 68. The bisectors of the base angles of an isosceles triangle are equal.

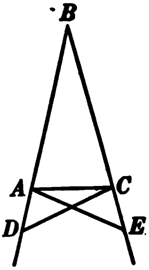
114. Historical Note. Exercise 69 is known as the *pons asinorum*, or bridge of asses, since it has proved difficult to many beginners in geometry. This proposition and the proof here suggested are due to Euclid, a great mathematician who wrote the first systematic text-book on geometry. In this work, known as Euclid's *Elements*, the exercise here given is the fifth proposition in Book I.

EUCLID

Of the life of Euclid there is but little known except that he was gentle and modest and "was a Greek who lived and taught in Alexandria about 300 B.C." To him is attributed the saying, "There is no royal road to geometry." His appreciation of the culture value of geometry is shown in a story related by Stobaeus (which is probably authentic). "A lad who had just begun geometry asked, 'What do I gain by learning all this stuff?' Euclid called his slave and said, 'Give this boy some coppers, since he must make a profit out of what he learns.'"

Ex. 69. By using the accompanying diagram prove that the base angles of an isosceles triangle are equal.

HINT. Prove $\triangle ABE = \triangle DBC$. Then prove $\triangle ACE = \triangle DAC$.



Ex. 70. (*a*) If equal segments measured from the vertex are laid off on the arms of an isosceles triangle, the lines drawn from the ends of the segments to the foot of the bisector of the vertex angle will be equal.

(*b*) Extend (*a*) to the case in which the equal segments are laid off on the arms prolonged through the vertex.

Ex. 71. (*a*) If equal segments measured from the ends of the base are laid off on the arms of an isosceles triangle, the lines drawn from the ends of the segments to the foot of the bisector of the vertex angle will be equal.

(*b*) Extend (*a*) to the case in which the equal segments are laid off on the arms prolonged through the ends of the base.

Ex. 72. (*a*) If equal segments measured from the ends of the base are laid off on the base of an isosceles triangle, the lines joining the vertex of the triangle to the ends of the segments will be equal.

(*b*) Extend (*a*) to the case in which the equal segments are laid off on the base prolonged (Fig. 1).

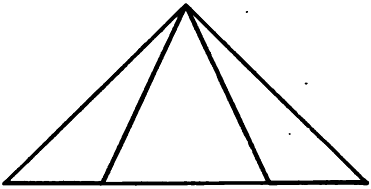


Fig. 1.

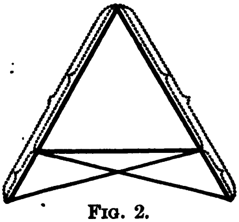
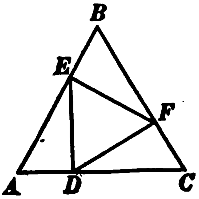


Fig. 2.

Ex. 73. (*a*) If equal segments measured from the ends of the base are laid off on the arms of an isosceles triangle, the lines drawn from the ends of the segments to the opposite ends of the base will be equal.

(*b*) Extend (*a*) to the case in which the equal segments are laid off on the arms prolonged through the ends of the base (Fig. 2).

Ex. 74. Triangle ABC is equilateral, and $AE = BF = CD$. Prove triangle EFD equilateral.



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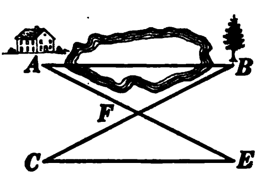
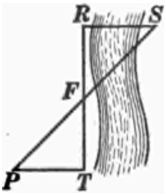
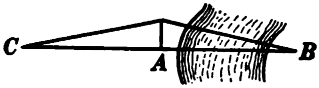
115. Measurement of Distances by Means of Triangles. The theorems which prove triangles equal are applied practically in measuring distances on the surface of the earth. Thus, if it is desired to find the distance between two places, A and B, which are separated by a pond or other obstruction, place a stake at some point accessible to both A and B, as F. Measure the distances FA and FB; then, keeping in line with F and B, measure CF equal to FB, and, in line with F and A, measure FE equal to FA. Lastly measure CE, and the distance from A to B is thus obtained, since AB is equal to CE. Can this method be used when A and B are on opposite sides of a hill and each is invisible from the other?



Ex. 75. Show how to find the distance across a river by taking the following measurements. Measure a convenient distance along the bank, as *RT*, and fix a stake at its mid-point, *F*. Proceed at right angles to *RT* from *T* to the point *P*, where *F*, *S*, and *P* are in line; measure *PT*.



Ex. 76. An army engineer wished to obtain quickly the approximate distance across a river, and had no instruments with which to make measurements. He stood on the bank of the river, as at *A*, and sighted the opposite bank, or *B*. Then without raising or lowering his eyes, he faced about, and his line of sight struck the ground at *C*. He paced the distance, *AC*, and gave this as the distance across the river. Explain his method.



Ex. 77. Tell what measurements to make to obtain the distance between two inaccessible points, *R* and *S* (Fig. 1).

Ex. 78. The fact that a triangle is determined if its base and its base angles are given was used as early as the time of Thales (640 B.C.) to find the distance of a ship at sea; the base of the triangle was usually a lighthouse tower and the base angles were found by observation. Draw a figure and explain.

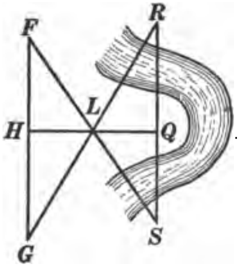


FIG. 1.

Ex. 79. Explain the following method of finding *SR* (Fig. 2). Place a stake at *S*, and another at a convenient point, *Q*, in line with *S* and *R*. From a convenient point, as *T*, measure *TS* and *TQ*. Prolong *QT*, and make *TF* equal to *QT*. Prolong *ST*, and make *TB* equal to *ST*. Then keep in line with *F* and *B*, until a point is reached, as *G*, where *T* and *R* come into line. Then *BG* is equal to the required distance, *RS*.

Ex. 80. In an equilateral triangle, if two lines are drawn from the ends of the base, making equal angles with the base, and terminated by the other two sides, the lines are equal.

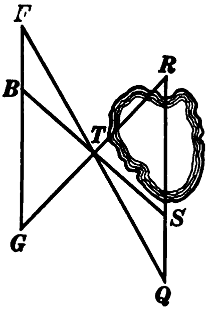
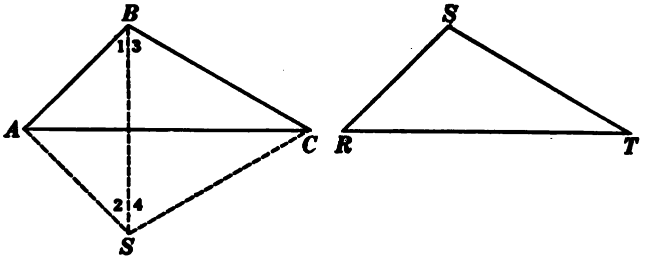


FIG. 2.

PROPOSITION V. THEOREM

116. *Two triangles are equal if the three sides of one are equal respectively to the three sides of the other.*



Given $\triangle s$ ABC and RST, $AB = RS$, $BC = ST$, and $CA = TR$.

To prove $\triangle ABC = \triangle RST$.

ARGUMENT	REASONS
1. Place $\triangle RST$ so that the longest side RT shall fall upon its equal AC, R upon A, T upon C, and so that S shall fall opposite B.	1. Any geometric figure may be moved from one position to another without change of size or shape. § 54, 14.
2. Draw BS.	2. A str. line may be drawn from any one point to any other. § 54, 15.
3. $\triangle ABS$ is isosceles.	3. $AB = RS$, by hyp.
4. $\therefore \angle 1 = \angle 2$.	4. The base $\angle s$ of an isosceles $\triangle$ are equal. § 111.
5. $\triangle BCS$ is isosceles.	5. $BC = ST$, by hyp.
6. $\therefore \angle 3 = \angle 4$.	6. Same reason as 4.
7. $\angle 1 + \angle 3 = \angle 2 + \angle 4$.	7. If equals are added to equals, the sums are equal. § 54, 2.

ARGUMENT	REASONS
8. $\therefore \angle ABC = \angle CSA$.	8. The whole = the sum of all its parts. § 54, 11.
9 $\therefore \triangle ABC = \triangle CSA$; *i.e.* $\triangle ABC = \triangle RST$. Q.E.D.	9. Two $\triangle$s are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other. § 107.

Ex. 81. (*a*) Prove Prop. V, using two obtuse triangles and applying the shortest side of one to the shortest side of the other.

(*b*) Prove Prop. V, using two right triangles and applying the shortest side of one to the shortest side of the other.

117. Question. Why is not Prop. V proved by superposition?

SUMMARY OF CONDITIONS FOR EQUALITY OF TRIANGLES

118. Two triangles are equal if
- a side and the two adjacent angles
- two sides and the included angle
- three sides

of one are equal respectively to
- a side and the two adjacent angles
- two sides and the included angle
- three sides

of the other.

Ex. 82. The median to the base of an isosceles triangle bisects the angle at the vertex and is perpendicular to the base.

Ex. 83. In a certain quadrilateral two adjacent sides are equal; the other two sides are also equal. Find a pair of triangles which you can prove equal.

Ex. 84. If the opposite sides of a quadrilateral are equal, the opposite angles also are equal.

Ex. 85. If two isosceles triangles have the same base, the line joining their vertices bisects each vertex angle and is perpendicular to the common base. (Two cases.)

Ex. 86. In what triangles are the three medians equal?

Ex. 87. In what triangles are two medians equal?

Ex. 88. If three rods of different lengths are put together to form a triangle, can a different triangle be formed by arranging the rods in a different order? Will the angles opposite the same rods always be the same?

Ex. 89. If two sides of one triangle are equal respectively to two sides of another, and the median drawn to one of these sides in the first is equal to the median drawn to the corresponding side in the second, the triangles are equal.

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119. Def. A **circle** is a plane closed figure whose boundary is a curve such that all straight lines to it from a fixed point within are equal.

The curve which forms the boundary of a circle is called the **circumference.** The fixed point within is called the **center,** and a line joining the center to any point on the circumference is a **radius.**

120. It follows from the definition of a circle that:

All radii of the same circle are equal.

121. Def. Any portion of a circumference is called an **arc.**

122. Assumption 18. Circle postulate. *A circle may be constructed having any point as center, and having a radius equal to any finite line.*

123. The *solution* of a *problem of construction* consists of three distinct steps:

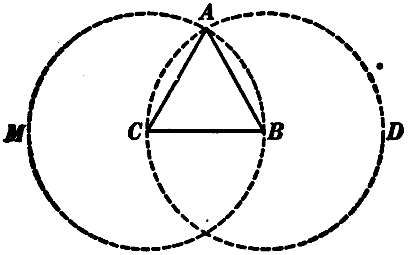
(1) *The construction, i.e.* the process of drawing the required figure with ruler and compasses.

(2) *The proof,* a demonstration that the figure constructed fulfills the given conditions.

(3) *The discussion, i.e.* a statement of the conditions under which there may be no solution, one solution, or more than one.

Proposition VI. Problem

124. *To construct an equilateral triangle, with a given line as side.*



Given line *BC*.

To construct an equilateral triangle on *BC*.

I. Construction

1. With *B* as center and *BC* as radius, construct circle *CAD*.
2. With *C* as center and *BC* as radius, construct circle *BMA*.
3. Connect point *A*, at which the circumferences intersect, with *B* and *C*.
4. △ *ABC* is the required triangle.

II. Proof

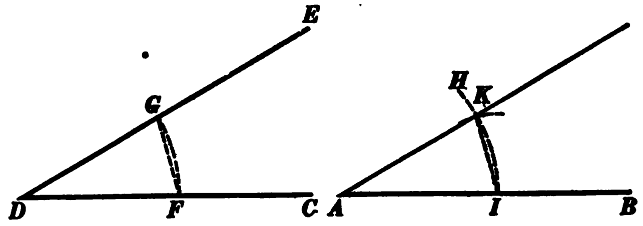
Argument	Reasons
1. $AB = BC$ and $CA = BC$.	1. All radii of the same circle are equal. § 120.
2. $\therefore AB = BC = CA$.	2. Things equal to the same thing are equal to each other. § 54, 1.
3. ∴ △ *ABC* is equilateral. Q.E.D.	3. A △ having its three sides equal is equilateral. § 95.

III. Discussion

This construction is always possible, and there is only one solution. (See § 116.)

PROPOSITION VII. PROBLEM

125. ***With a given vertex and a given side, to construct an angle equal to a given angle.***



Given vertex A, side AB, and $\angle CDE$.

To construct an $\angle$ equal to $\angle CDE$ and having A as vertex and AB as side.

I. Construction

1. With D as center, and with any convenient radius, describe an arc intersecting the sides of $\angle D$ at F and G, respectively.

2. With A as center, and with the same radius, describe the indefinite arc IH, cutting AB at I.

3. With I as center, and with a radius equal to str. line FG, describe an arc intersecting the arc IH at K.

4. Draw AK.

5. $\angle BAK = \angle CDE$, and is the $\angle$ required.

II. Proof

ARGUMENT	REASONS
1. Draw FG and IK.	1. A str. line may be drawn from any one point to any other. § 54, 15.
2. In $\triangle$ FDG and IAK, $DF = AI$.	2. By cons.
3. $DG = AK$.	3. By cons.
4. $FG = IK$.	4. By cons.

Argument	Reasons
5. $\therefore \triangle FDG = \triangle IAK$.	5. Two $\triangle$ are equal if the three sides. of one are equal respectively to the three sides of the other. § 116.
6. $\therefore \angle BAK = \angle CDE$. Q.E.D.	6. Homol. parts of equal figures are equal. § 110.

III. **Discussion**

This construction is always possible, and there is only one solution.

Ex. 90. Construct a triangle, given two sides and the included angle.

Ex. 91. Construct an isosceles triangle, given the vertex angle and an arm.

Ex. 92. Construct a triangle, given a side and the two adjacent angles.

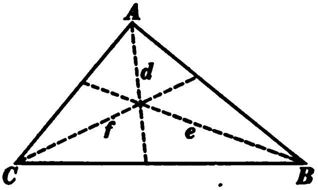
Ex. 93. Construct an isosceles triangle, given an arm and one of the equal angles.

Ex. 94. How many parts determine a triangle? Do three angles determine it? Explain.

Ex. 95. Construct an isosceles triangle, given the base and an arm.

Ex. 96. Construct a scalene triangle, given the three sides.

126. Def. The **bisector of an angle of a triangle** is the line from the vertex of the angle bisecting the angle and limited by the opposite side of the triangle.



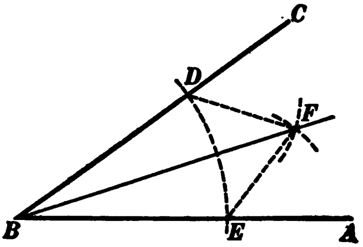
Ex. 97. In what triangles are the three bisectors equal?

Ex. 98. In what triangles are two bisectors, and only two, equal?

Ex. 99. In what triangles are the medians. the bisectors, and the altitudes identical?

Proposition VIII. Problem

127. *To construct the bisector of a given angle.*



Given $\angle ABC$.

To construct the bisector of $\angle ABC$.

I. Construction

1. With B as center, and with any convenient radius describe an arc intersecting BA at E and BC at D.

2. With D and E as centers, and with equal radii, describe arcs intersecting at F.

3. Draw BF.

4. BF is the bisector of $\angle ABC$.

II. Proof

Argument	Reasons
1. In $\triangle$s EBF and FBD, $BE = BD$.	1. By cons.
2. $EF = DF$.	2. By cons.
3. $BF = BF$.	3. By iden.
4. $\therefore \triangle EBF = \triangle FBD$.	4. Two $\triangle$s are equal if the three sides of one are equal respectively to the three sides of the other. § 116.
5. $\therefore \angle EBF = \angle FBD$.	5. Homol. parts of equal figures are equal. § 110

ARGUMENT	REASONS
6. ∴ BF is the bisector of $\angle ABC$. Q.E.D.	6. The bisector of an ∠ is the line which divides the ∠ into two equal ∠s. § 53.

III. Discussion

This construction is always possible, and there is only one solution.

Ex. 100. Draw an obtuse angle and divide it into: (*a*) four equal angles; (*b*) eight equal angles.

Ex. 101. Construct the bisector of the vertex angle of an isosceles triangle.

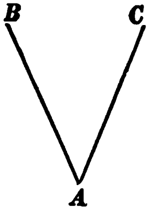
Ex. 102. Draw two intersecting lines and construct the bisectors of the four angles formed.

Ex. 103. Bisect an angle between two bisectors in Ex. 102, and find the number of degrees in each angle.

Ex. 104. Construct the bisector of an exterior angle at the base of an isosceles triangle.

Ex. 105. Construct the bisectors of the three angles of any triangle. What can you infer about them? Can the correctness of this inference be proved by making a careful construction?

128. Def. The **distance** between two points is the length of the straight line joining them. Thus if three points, A, B, and C, are so located that $AB = AC$, A is said to be **equidistant** from B and C.



Ex. 106. Find all the points on the blackboard which are one foot from a fixed point, P, on the blackboard.

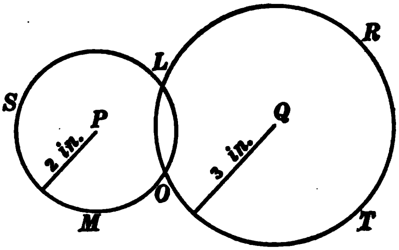
Ex. 107. Draw a line, AB, on the blackboard and mark some point near the line, as P. Find all the points in AB that are a foot from P.

Ex. 108. Mark a point, Q, on the blackboard. Find all the points on the blackboard which are: (*a*) ten inches from Q; (*b*) four inches from Q. How far are the points of (*b*) from the points of (*a*), if the distance is measured on a line through Q?

LOCI

129. In many geometric problems it is necessary to *locate* all points which satisfy certain prescribed conditions, or to determine the path traced by a point which moves according to certain fixed laws. Thus, the points in a plane two inches from a given point are in the circumference of a circle whose center is the given point and whose radius is two inches.

Again, let it be required to find all points in a plane two inches from one fixed point and three inches from another. All points two inches from the fixed point P are in the circumference of the circle LMS, having P for center and having a radius equal to two inches. All points three inches from the fixed point Q are in the circumference of the circle LRT, having Q for center and having a radius equal to three inches. If the two circles are wholly outside of each other, there will be *no* points satisfying the two prescribed conditions; if the two circumferences touch, but do not intersect, there will be *one* point; if the two circumferences intersect, there will be *two* points. It will be proved later (§ 324) that there cannot be more than two points which satisfy both of the given conditions.



130. Def. A figure is the **locus** of all points which satisfy one or more given conditions, if all points in the figure satisfy the given conditions and if these conditions are satisfied by no other points.

A **locus**, then, is an *assemblage of points* which obey one or more definite laws.

It is often convenient to locate these points by thinking of them as *the path* traced by a moving point the motion of which is controlled by certain fixed laws.

131. In plane geometry a *locus* may be composed of one or more points or of one or more lines, or of any combination of points and lines.

132. Questions.* — What is the locus of all points in space two inches from a given point? What is the locus of all points in space two inches from a given plane? What is the locus of all points in space such that perpendiculars from them to a given plane shall be equal to a given line? What is the locus of all points on the surface of the earth midway between the north and south poles? 23½° from the equator? 23½° from the north pole? 90° from the equator? What is the locus of a gas jet four feet from the ceiling of this room? four feet from the ceiling and five feet from a side wall? four feet from the ceiling, five feet from a side wall, and six feet from an end wall?

Ex. 109. Given an *unlimited line* AB and a point P. Find all points in AB which are also: (*a*) three inches from P; (*b*) at a given distance, a, from P.

Ex. 110. Given a circle with center O and radius six inches. State, without proof, the locus: (*a*) of all points four inches from O; (*b*) of all points five inches from the circumference of the circle, measured on the radius or radius prolonged.

Ex. 111. Given the base and one adjacent angle of a triangle, what is the locus of the vertex of the angle opposite the base? (State without proof.)

Ex. 112. Given the base and one other side of a triangle, what is the locus of the vertex of the angle opposite the base? (State without proof.)

Ex. 113. Given the base and the other two sides of a triangle, what is the locus of the vertex of the angle opposite the base?

Ex. 114. Given the base of a triangle and the median to the base, what is the locus of the end of the median which is remote from the base?

Ex. 115. Given the base of a triangle, one other side, and the median to the base, what is the locus of the vertex of the angle opposite the base?

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133. Question. In which of the exercises above was a triangle ***determined?***

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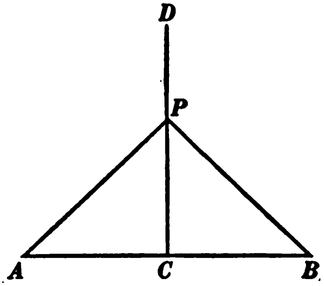
* In order to develop the imagination of the student the authors deem it advisable in this article to introduce questions involving loci in space. It should be noted that no proofs of answers to these questions are demanded.

PROPOSITION IX. THEOREM

134. *Every point in the perpendicular bisector of a line is equidistant from the ends of that line.*



Given line AB, its $\perp$ bisector CD, and P any point in CD.

To prove $PA = PB$.

ARGUMENT	REASONS
1. In $\triangle$ APC and CPB, $AC = CB$.	1. By hyp.
2. $PC = PC$.	2. By iden.
3. $\angle PCA = \angle BCP$.	3. All rt. $\angle$ are equal. § 64.
4. $\therefore \triangle APC = \triangle CPB$.	4. Two $\triangle$ are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other. § 107.
5. $\therefore PA = PB$. Q. E. D.	5. Homol. parts of equal figures are equal. § 110.

Ex. 116. Four villages are so located that B is 25 miles east of A, C 20 miles north of A, and D 20 miles south of A. Prove that B is as far from C as it is from D.

Ex. 117. In a given circumference, find the points equidistant from two given points, A and B.

135. Def. One theorem is the **converse** of another when the conclusion of the first is the hypothesis of the second, and the hypothesis of the first is the conclusion of the second.

The converse of a truth is not always true; thus, "*All men are bipeds*" is true, but the converse, "*All bipeds are men,*" is false. "*All right angles are equal*" is true, but "*All equal angles are right angles*" is false.

136. Def. One theorem is the **opposite** of another when the hypothesis of the first is the contradiction of the hypothesis of the second, and the conclusion of the first is the contradiction of the conclusion of the second.

The opposite of a truth is not always true; thus, "*If a man lives in the city of New York, he lives in New York State,*" is true, but the opposite, "*If a man does not live in the city of New York, he does not live in New York State,*" is false.

137. Note. If the converse of a proposition is true, the opposite also is true; so, too, if the opposite of a proposition is true, the converse also is true.

This may be evident to the student after a consideration of the following type forms:

(1) DIRECT	(2) CONVERSE	(3) OPPOSITE
If A is B,	If C is D,	If A is not B,
Then C is D.	Then A is B.	Then C is not D.

If (2) is true, then (3) must be true. Again, if (3) is true, then (2) must be true.

138. A necessary and sufficient *test* of the *completeness* of a definition is that its converse shall also be true. Hence a definition may be quoted as the reason for a converse or for an opposite as well as for a direct statement in an argument.

Ex. 118. State the converse of the definition for equal figures; straight line; plane surface.

Ex. 119. State the converse of: If one straight line meets another straight line, the sum of the two adjacent angles is two right angles.

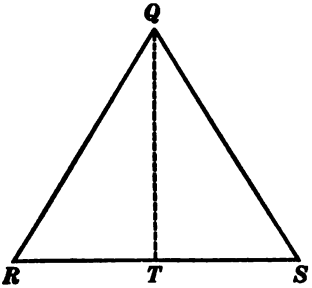
Ex. 120. State the converse and opposite of Prop. IX.

Ex. 121. State the converse of Prop. I. Is it true?

Proposition X. Theorem

(Converse of Prop. IX)

139. *Every point equidistant from the ends of a line lies in the perpendicular bisector of that line.*



Given line RS, and point Q such that $QR = QS$.

To prove that Q lies in the $\perp$ bisector of RS.

Argument	Reasons
1. Let QT bisect $\angle RQS$.	1. Every $\angle$ has but one bisector. § 53.
2. $QR = QS$.	2. By hyp.
3. $\therefore \triangle RQS$ is isosceles.	3. A $\triangle$ having two sides equal is an isosceles $\triangle$. § 94.
4. $\therefore QT$ is the $\perp$ bisector of RS.	4. The bisector of the $\angle$ at the vertex of an isosceles $\triangle$ is $\perp$ to the base and bisects it. § 112.
5. $\therefore Q$ lies in the $\perp$ bisector of RS. Q.E.D.	5. By proof.

140. Cor. I. *Every point not in the perpendicular bisector of a line is not equidistant from the ends of the line.*

Hint. Use § 137, or contradict the conclusion and tell why the contradiction is false.

141. Cor. II. *The locus of all points equidistant from the ends of a given line is the perpendicular bisector of that line.*

(For model proofs of locus theorems, see pp. 297 and 298.)

142. Cor. III. *Two points each equidistant from the ends of a line determine the perpendicular bisector of the line.*

Hint. Use § 139 and § 25.

143. In order to prove that a locus problem is *solved* it is *necessary and sufficient* to show two things:

(1) That every point *in* the proposed locus satisfies the prescribed conditions.

(2) That every point *outside* of the proposed locus does not satisfy the prescribed conditions.

Instead of proving (2), it may frequently be more convenient to prove:

(2′) That every point which satisfies the prescribed conditions lies in the proposed locus.

144. Note. In exercises in which the student is asked to "Find a locus," it must be understood that he has not *found* a locus until he has given a *proof* with regard to it as outlined above. The proof must be based upon a *direct proposition* and its *opposite*; or, upon a *direct proposition* and its *converse*.

Ex. 122. Find the locus of all points equidistant from two given points A and B.

Ex. 123. In a given *unlimited line* AB, find a point equidistant from two given points C and D not on this line.

Ex. 124. Given a circle with center O, also a point P. Find all points which lie in the circumference of circle O, and which are also (*a*) two inches from P; (*b*) a distance of d from P.

Ex. 125. Find all points at a distance of d from a given point P, and at the same time at a distance of m from a given point Q.

Ex. 126. Given a circle O with radius r. Find the locus of the midpoints of the radii of the circle.

Ex. 127. Given two circles having the same center. State, without proof, the locus of a point equidistant from their circumferences.

Ex. 128. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

PROPOSITION XI. PROBLEM

145. *To construct the perpendicular bisector of a given straight line.*

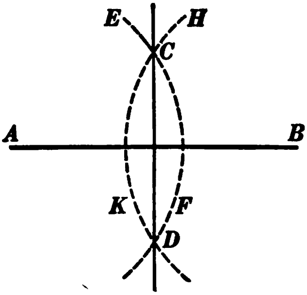


FIG. 1.

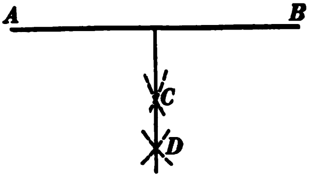


FIG. 2.

Given line AB (Fig. 1).

To construct the perpendicular bisector of AB.

I. Construction

1. With A as center, and with a convenient radius greater than half AB, describe the arc EF.

2. With B as center, and with the same radius, describe the arc HK.

3. Let C and D be the points of intersection of these two arcs.

4. Connect points C and D.

5. CD is the ⊥ bisector of AB.

II. The proof and discussion are left as an exercise for the student.

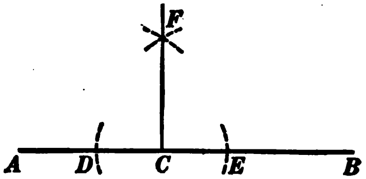
HINT. Apply § 142.

146. Note. The construction given in Fig. 2 may be used when the position of the given line makes it more convenient.

147. Question. Is it necessary that CA shall equal CB? that CA shall equal DA (Fig. 1)? Give the equations that *must* hold.

PROPOSITION XII. PROBLEM

148. *To construct a perpendicular to a given straight line at a given point in the line.*



Given line AB and point C in the line.

To construct a ⊥ to AB at C.

I. Construction

1. With C as center, and with any convenient radius, draw arcs cutting AB on each side of C, as at D and E.

2. Then with D and E as centers, and with a longer radius, draw two arcs intersecting each other at F.

3. Draw FC.

4. FC is ⊥ to AB at C.

II. The proof and discussion are left as an exercise for the student.

Ex. 129. Construct the perpendicular bisector of a line given at the bottom of a page or of a blackboard.

Ex. 130. Divide a given line into four equal parts.

Ex. 131. Construct the three medians of a triangle.

Ex. 132. Construct a perpendicular to a line at a given point when the given point is one end of the line. (HINT. Prolong the line.)

Ex. 133. Construct a right triangle, given the two arms a and b.

Ex. 134. Construct a right triangle, given the hypotenuse and an arm.

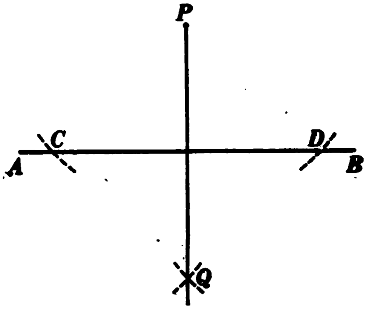
Ex. 135. Construct the complement of a given angle.

Ex. 136. Construct an angle of 45°; of 135°.

Ex. 137. Construct a quadrilateral, given four sides and the angle between two of them.

PROPOSITION XIII. PROBLEM

149. ***From a point outside a line to construct a perpendicular to the line.***



Given line AB and point P outside of AB.

To construct a ⊥ from P to AB.

I. Construction

1. With P as center, and with a radius of sufficient length, describe an arc cutting AB at points C and D.

2. With C and D as centers, and with any convenient radius, describe arcs intersecting at Q.

3. Draw PQ.

4. PQ is a ⊥ from P to AB.

II. The proof is left as an exercise for the student. The discussion will be given in § 154.

150. Question. Must P and Q be on opposite sides of AB? Is it necessary that $PC = QC$?

Ex. 138. Construct the three altitudes of an acute triangle. Do they seem to meet? where?

Ex. 139. Construct the three altitudes of a right triangle. Where do they seem to meet?

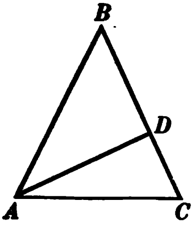
Ex. 140. Construct the three altitudes of an obtuse triangle. Where do they seem to meet?

Ex. 141. Construct a triangle ABC, given two sides and the median drawn to one of them. Abbreviate thus: given a, b, and m_a.

151. Analysis of a problem of construction. In the more difficult problems of construction a course of reasoning is sometimes necessary to enable the student to discover the process of drawing the required figure. This course of reasoning is called the **analysis** of the problem. It is illustrated in § 152 and is more fully treated in the exercises following § 274.

152. Note. In such problems as Ex. 141, it is well first to imagine the problem solved and to sketch a figure to represent the desired construction. Then mark (with colored crayon, if convenient) the parts supposed to be given. By studying carefully the relation of the given parts to the whole figure, try to find some part of the figure that you can construct. This will generally be a *triangle.* After this part is constructed it is usually an easy matter to complete the required figure.

Thus: Problem. Let it be required to construct an isosceles triangle, given an arm and the altitude upon it. By studying the figure with the given parts marked (heavy or with colored crayon), it will be seen that the solution of the problem depends in this case upon the construction of a right triangle, given the hypotenuse and one arm. The right triangle ABD may now be constructed, and it will be readily seen that to complete the construction it is only necessary to prolong BD to C, making $BC = AB$, and to connect A and C.



Ex. 142. Construct a triangle ABC, given two sides and an altitude to one of the given sides. Abbreviate thus: given a, b, and h_b.

Ex. 143. Construct a triangle ABC, given a side, an adjacent angle, and the altitude to the side opposite the given angle. Abbreviate thus: given a, B, and h_b.

Ex. 144. Construct an isosceles triangle, given an arm and the median to it.

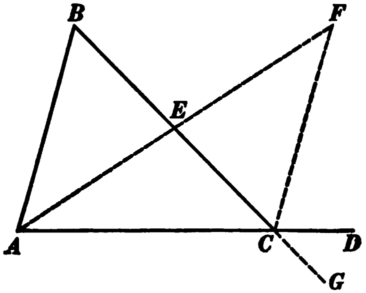
Ex. 145. Construct an isosceles triangle, given an arm and the angle which the median to it makes with it.

Ex. 146. Construct a triangle, given two sides and the angle which a median to one side makes: (*a*) with that side; (*b*) with the other side.

Ex. 147. Construct a triangle, given a side, an adjacent angle, and its bisector.

PROPOSITION XIV. THEOREM

153. *If one side of a triangle is prolonged, the exterior angle formed is greater than either of the remote interior angles.*



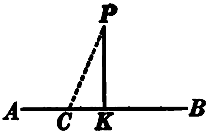
Given $\triangle ABC$ with AC prolonged to D, making exterior $\angle DCB$.

To prove $\angle DCB > \angle ABC$ or $\angle CAB$.

ARGUMENT	REASONS
1 Let E be the mid-point of BC; draw AE, and prolong it to F, making $EF = AE$. Draw CF.	1. A str. line may be drawn from any one point to any other. § 54, 15.
2. In $\triangle$s ABE and EFC, $BE = EC$.	2. By cons. E is the mid-point of BC.
3. $AE = EF$.	3. By cons.
4. $\angle BEA = \angle CEF$.	4. If two str. lines intersect, the vertical $\angle$s are equal. § 77.
5. $\therefore \triangle ABE = \triangle EFC$.	5. Two $\triangle$s are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other. § 107.
6. $\therefore \angle B = \angle FCE$.	6. Homol. parts of equal figures are equal. § 110.

ARGUMENT	REASONS
7. $\angle DCB > \angle FCE$.	7. The whole > any of its parts. § 54, 12.
8. $\therefore \angle DCB > \angle B$.	8. Substituting $\angle B$ for its equal $\angle FCE$.
9. Likewise, if BC is prolonged to G, $\angle ACG > \angle CAB$.	9. By bisecting line AC, and by steps similar to 1–8.
10. But $\angle DCB = \angle ACG$.	10. Same reason as 4.
11. $\therefore \angle DCB > \angle CAB$.	11. Substituting $\angle DCB$ for its equal $\angle ACG$.
12. $\therefore \angle DCB > \angle ABC$ or $\angle CAB$. Q.E.D.	12. By proof.

154. Cor. *From a point outside a line there exists only one perpendicular to the line.*



HINT. If there exists a second ⊥ to AB from P, as PC, then $\angle PCA$ and $\angle PKA$ are both rt. ∠s and are therefore equal. But this is impossible by § 153.

155. §§ 149 and 154 may be combined in one statement as follows:

From a point outside a line there exists one and only one perpendicular to the line.

Ex. 148. A triangle cannot contain two right angles.

Ex. 149. In the figure of Prop. XIV, is angle DCB necessarily greater than angle BCA? than angle B?

Ex. 150. In Fig. 1, prove that:

(1) Angle 1 is greater than angle CAE or angle AEC;

(2) Angle 5 is greater than angle CBA or angle BAE;

(3) Angle EDA is greater than angle 3;

(4) Angle 4 is greater than angle DAE.

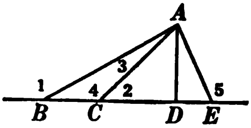


FIG. 1.

Ex. 151. In Fig. 2, show that angle 6 is greater than angle 7; also that angle 9 is greater than angle A.

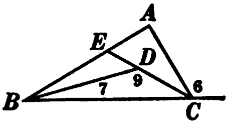
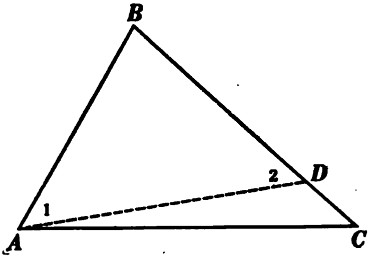


FIG. 2.

PROPOSITION XV. THEOREM

156. *If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.*



Given $\triangle ABC$ with $BC > BA$.

To prove $\angle CAB > \angle C$.

ARGUMENT	REASONS
1. On BC lay off $BD = AB$.	1. Circle post. §§ 122, 157.
2. Draw AD.	2. Str. line post. I. § 54, 15.
3. Then $\angle 1 = \angle 2$.	3. The base $\angle$s of an isosceles $\triangle$ are equal. § 111.
4. Now $\angle 2 > \angle C$.	4. If one side of a $\triangle$ is prolonged, the ext. $\angle$ formed $>$ either of the remote int. $\angle$s. § 153.
5. $\therefore \angle 1 > \angle C$.	5. Substituting $\angle 1$ for its equal $\angle 2$.
6. But $\angle CAB > \angle 1$.	6. The whole $>$ any of its parts. § 54, 12.
7. $\therefore \angle CAB > \angle C$. Q.E.D.	7. If three magnitudes of the same kind are so related that the first $>$ the second and the second $>$ the third, then the first $>$ the third. § 54, 10.

157. Note. Hereafter the student will not be required to state postulates and definitions in full unless requested to do so by the teacher.

158. Note. When two magnitudes are given unequal, the laying off of the less upon the greater will often serve as the initial step in developing a proof.

Ex. 152. Given the isosceles triangle RST, with ST the base and RT prolonged any length, as to K. Prove angle KSR greater than angle K.

Ex. 153. If two adjacent sides of a quadrilateral are greater respectively than the other two sides, the angle included between the two shorter sides is greater than the angle between the two greater sides.

Ex. 154. If from a point within a triangle lines are drawn to the ends of one of its sides, the angle between these lines is greater than the angle between the other two sides of the triangle. (See Ex. 151.)

---

159. The **indirect method**, or proof **by exclusion**, consists in contradicting the conclusion of a proposition, then showing the contradiction to be false. The conclusion of the proposition is thus established. This process requires an examination of *every possible contradiction* of the conclusion. For example, to prove indirectly that A equals B it would be necessary to consider the only three suppositions that are admissible in this case, viz.:

(1) $A > B$,
(2) $A < B$,
(3) $A = B$.

By proving (1) and (2) false, the truth of (3) is established, *i.e.* $A = B$. This method of reasoning is called *reductio ad absurdum.* It enables us to establish a conclusion by showing that every contradiction of it leads to an absurdity. Props. XVI and XVII will be proved by the indirect method.

160. Question. Would it be possible to base a proof upon a contradiction of the hypothesis?

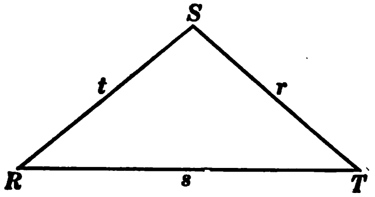
161. (*a*) In the use of the indirect method the student should give, as argument 1, all the suppositions of which the case he is considering admits, including the conclusion. As reason 1 the number of such possible suppositions should be cited.

(*b*) As a reason for the last step in the argument he should state which of these suppositions have been proved false.

PROPOSITION XVI. THEOREM

(Converse of Prop. IV)

162. *If two angles of a triangle are equal, the sides opposite are equal.*



Given $\triangle RST$ with $\angle R = \angle T$.

To prove $r = t$.

ARGUMENT	REASONS
1. $r > t$, $r < t$, or $r = t$.	1. In this case only three suppositions are admissible.
2. First suppose $r > t$; then $\angle R > \angle T$.	2. If two sides of a $\triangle$ are unequal, the $\angle$ opposite the greater side $>$ the $\angle$ opposite the less side. § 156.
3. This is impossible.	3. By hyp., $\angle R = \angle T$.
4. Next suppose $r < t$; then $\angle R < \angle T$.	4. Same reason as 2.
5. This is impossible.	5. Same reason as 3.
6. $\therefore r = t$.	6. The two suppositions, $r > t$ and $r < t$, have been proved false.
Q.E.D.	

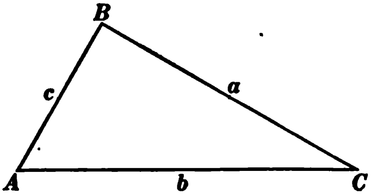
163. Cor. *An equiangular triangle is also equilateral.*

Ex. 155. The bisectors of the base angles of an isosceles triangle form an isosceles triangle.

PROPOSITION XVII. THEOREM

(Converse of Prop. XV)

164. *If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.*



Given $\triangle ABC$ with $\angle A > \angle C$.

To prove $a > c$.

ARGUMENT	REASONS
1. $a < c$, $a = c$, or $a > c$.	1. In this case only three suppositions are admissible.
2. First suppose $a < c$; then $\angle A < \angle C$.	2. If two sides of a $\triangle$ are unequal, the $\angle$ opposite the greater side > the $\angle$ opposite the less side. § 156.
3. This is impossible.	3. By hyp., $\angle A > \angle C$.
4. Next suppose $a = c$; then $\angle A = \angle C$.	4. The base $\angle$s of an isosceles $\triangle$ are equal. § 111.
5. This is impossible.	5. Same reason as 3.
6. $\therefore a > c$.	6. The two suppositions, $a < c$ and $a = c$, have been proved false.
Q.E.D.	

165. Cor. *The perpendicular is the shortest straight line from a point to a line.*

166. Def. The length of the perpendicular from a point to a line is called the **distance** from the point to the line.

Ex. 156. The sum of the altitudes of any triangle is less than the perimeter of the triangle.

Ex. 157. Given the quadrilateral $ABCD$ with B and D right angles and BC greater than CD. Prove AD greater than AB.

Ex. 158. Given triangle ABC, with $AC > BC$. Let bisectors of angles A and B meet at O. Prove $AO > BO$.

Ex. 159. A line drawn from the vertex of an isosceles triangle to any point in the base is less than one of the equal sides of the triangle.

Ex. 160. If ABC and ABD are two triangles on the same base and on the same side of it such that $AC = BD$ and $AD = BC$, and if AC and BD intersect at O, prove triangle AOB isosceles.

Ex. 161. Prove Prop. XVI by using the figure and method of Ex. 69.

Ex. 162. Upon a given base is constructed a triangle one of the base angles of which is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

Ex. 163. If the four sides of a quadrilateral are equal, its diagonals bisect each other.

Ex. 164. The diagonals of an equilateral quadrilateral are perpendicular to each other, and they bisect the angles of the quadrilateral.

Ex. 165. If two adjacent sides of a quadrilateral are equal and the other two sides are equal, one diagonal is the perpendicular bisector of the other. Tell which one is the bisector and prove the correctness of your answer.

Ex. 166. If, from a point in a perpendicular to a line, oblique lines are drawn cutting off equal segments from the foot of the perpendicular, the oblique lines are equal.

Ex. 167. State and prove the converse of Ex. 166.

Ex. 168. If, from a point in a perpendicular to a line, oblique lines are drawn cutting off unequal segments from the foot of the perpendicular, the oblique lines are unequal. Prove by laying off the less segment upon the greater. Then use Ex. 166, § 153, and § 164.

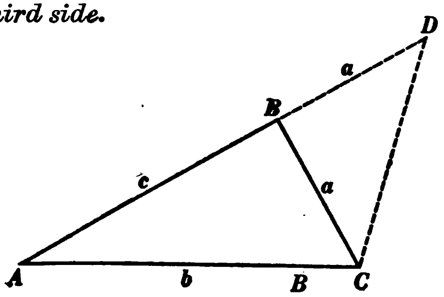
Ex. 169. If, from any point in a perpendicular to a line, two unequal oblique lines are drawn to the line, the oblique lines will cut off unequal segments from the foot of the perpendicular. Prove by the indirect method.

Ex. 170. By means of Prop. XIV, prove that the sum of any two angles of a triangle is less than two right angles.

Ex. 171. Construct a triangle ABC, given two sides, a and b, and the altitude to the third side, c. (See § 152.)

Proposition XVIII. Theorem

167. *The sum of any two sides of a triangle is greater than the third side.*



Given $\triangle ABC$.
To prove $a + c > b$.

Argument	Reasons
1. Prolong c through B until prolongation $BD = a$.	1. Str. line post. II. § 54, 16
2. Draw CD.	2. Str. line post. I. § 54, 15
3. In isosceles $\triangle BDC$, $\angle D = \angle DCB$.	3. The base ∠s of an isosceles $\triangle$ are equal. § 111.
4. But $\angle DCA > \angle DCB$.	4. The whole $>$ any of its parts. § 54, 12.
5. $\therefore \angle DCA > \angle D$.	5. Substituting $\angle D$ for its equal, $\angle DCB$.
6. $\therefore$ in $\triangle ADC$, $AD > b$.	6. If two ∠s of a $\triangle$ are unequal, the side opposite the greater $\angle$ is $>$ the side opposite the less angle. § 164.
7. $\therefore a + c > b$. Q.E.D.	7. Substituting $a + c$ for AD.

168. Cor. I. *Any side of a triangle is less than the sum and greater than the difference of the other two.*

169. Cor. II. *Any straight line is less than the sum of the parts of a broken line having the same extremities.*

170. Note to Teacher. Teachers who prefer to assume that "a straight line is the shortest line between two points" may omit Prop. XVIII entirely. Then Prop. XVII may be proved by a method similar to that used in Prop. XV. (See Ex. 172.)

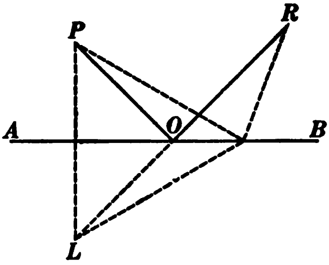
Ex. 172. Prove Prop. XVII by using the hint contained in § 158.

Ex. 173. If two sides of a triangle are 14 and 9, between what limiting values must the third side be?

Ex. 174. If the opposite ends of any two non-intersecting line segments are joined, the sum of the joining lines is greater than the sum of the other two lines.

Ex. 175. Given two points, P and R, and a line AB not passing through either. To find a point O, on AB, such that $PO + OR$ shall be as small as possible.

This exercise illustrates the law by which light is reflected from a mirror. The light from the object, P, is reflected and appears to come from L, as far behind the mirror as P is in front of it.



Ex. 176. If from any point within a triangle lines are drawn to the extremities of any side of the triangle, the sum of these lines is less than the sum of the other two sides of the triangle.

HINT. Let ABC be the given triangle, D the point within. Prolong AD until it intersects BC at E. Apply Prop. XVIII.

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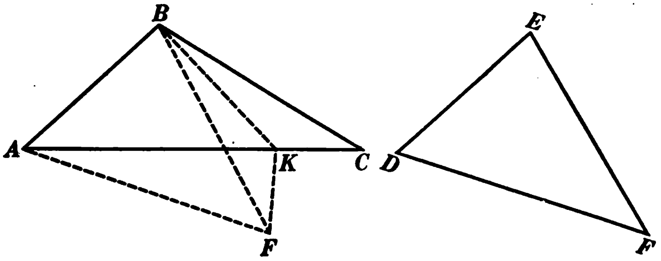
171. Note to Teacher. Up to this point all proofs given have been complete, including argument and reasons. In written work it is frequently convenient, however, to have students give the argument only. These two forms will be distinguished by calling the former a **complete demonstration** and the latter, which is illustrated in Prop. XIX, **argument only.**

It is often a sufficient test of a student's understanding of a theorem to have him state merely the main points involved in a proof. This may be given in enumerated steps, as in Prop. XXXV, or in the form of a paragraph, as in Prop. XLIV. This form will be called **outline of proof.**

PROPOSITION XIX. THEOREM

172. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given two $\triangle$ ABC and DEF with $AB = DE$, $BC = EF$, but $\angle ABC > \angle E$.

To prove $AC > DF$.

ARGUMENT ONLY

1. Place $\triangle DEF$ on $\triangle ABC$ so that DE shall fall upon its equal AB, D upon A, E upon B.
2. EF will then fall between AB and BC. Denote $\triangle DEF$ in its new position by ABF.
3. Draw BK bisecting $\angle FBC$ and meeting AC at K.
4. Draw KF.
5. In $\triangle$ FBK and KBC, $BC = BF$.
6. $BK = BK$.
7. $\angle FBK = \angle KBC$.
8. $\therefore \triangle FBK = \triangle KBC$.
9. $\therefore KF = KC$.
10. $AK + KF > AF$.
11. $\therefore AK + KC > AF$.
12. That is, $AC > AF$.
13. $\therefore AC > DF$. Q.E.D.

Ex. 177. (*a*) Draw a figure and discuss the case for Prop. XIX when F falls on AC; when F falls within triangle ABC. (*b*) Discuss Prop. XIX, taking AC (the base) $= DF$, $AB = DE$, and angle CAB greater than angle D.

Ex. 178. Prove Prop. XIX by using Fig. 1.

Hint. In $\triangle ACE$, $\angle CEA > \angle BEA$.

$\therefore \angle CEA > \angle EAB > \angle EAC$.

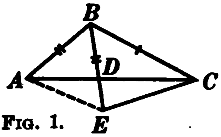
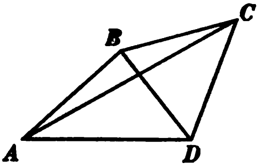


Fig. 1.

Ex. 179. Given triangle ABC with AB greater than BC, and let point P be taken on AB and point Q on CB, so that $AP = CQ$. Prove AQ greater than CP.

Ex. 180. Would the conclusion of Ex. 179 be true if P were taken on AB prolonged and Q on CB prolonged? Prove.

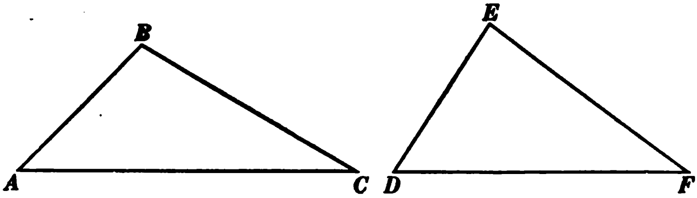
Ex. 181. In quadrilateral $ABCD$ if $AB = CD$, and angle CDA is greater than angle DAB, prove AC greater than BD.



Proposition XX. Theorem

(Converse of Prop. XIX)

173. *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given $\triangle$ ABC and DEF with $AB = DE$, $BC = EF$, but $AC > DF$

To prove $\angle B > \angle E$.

ARGUMENT	REASONS
1. $\angle B < \angle E$, $\angle B = \angle E$, or $\angle B > \angle E$.	1. In this case only three suppositions are admissible.
2. First suppose $\angle B < \angle E$; then $AC < DF$.	2. If two $\triangle$ have two sides of one equal respectively to two sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, then the third side of the first $>$ the third side of the second. § 172.
3. This is impossible.	3. By hyp., $AC > DF$.
4. Next suppose $\angle B = \angle E$; then $\triangle ABC = \triangle DEF$.	4. Two $\triangle$ are equal if two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other. § 107.
5. $\therefore AC = DF$.	5. Homol. parts of equal figures are equal. § 110.
6. This is impossible.	6. Same reason as 3.
7. $\therefore \angle B > \angle E$. Q.E.D.	7. The two suppositions, $\angle B < \angle E$ and $\angle B = \angle E$, have been proved false.

Ex. 182. If two opposite sides of a quadrilateral are equal, but its diagonals are unequal, then one angle opposite the greater diagonal is greater than one angle opposite the less diagonal.

Ex. 183. If two sides of a triangle are unequal, the median drawn to the third side makes unequal angles with the third side.

Ex. 184. If from the vertex S of an isosceles triangle BST a line is drawn to point P in the base BT so that BP is greater than PT, then angle BSP is greater than angle PST.

Ex. 185. If one angle of a triangle is equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

SUMMARY OF THEOREMS FOR PROVING ANGLES UNEQUAL

174. (*a*) *When the angles are in the same triangle:*

If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the less side.

(*b*) *When the angles are in different triangles:*

If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

(*c*) An exterior angle of a triangle is greater than either remote interior angle.

SUMMARY OF THEOREMS FOR PROVING LINES UNEQUAL

175. (*a*) *When the lines are in the same triangle:*

The sum of any two sides of a triangle is greater than the third side.

Any side of a triangle is less than the sum and greater than the difference of the other two.

If two angles of a triangle are unequal, the side opposite the greater angle is greater than the side opposite the less angle.

(*b*) *When the lines are in different triangles:*

If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

(*c*) Every point not in the perpendicular bisector of a line is not equidistant from the ends of the line.

The perpendicular is the shortest straight line from a point to a line.

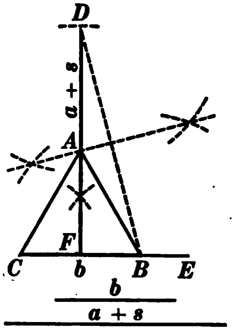
Any straight line is less than the sum of the parts of a broken line having the same extremities.

In some texts Exs. 168 and 169 are given as theorems.

176. Note. The student should note that the proof of Prop. XIX depends upon the device of substituting a broken line for an equal straight line. This method is further illustrated by the following exercises.

Ex. 186. Prove by the method discussed above that lines drawn to the ends of a line from any point not in its perpendicular bisector are unequal.

Ex. 187. Construct an isosceles triangle, given the base and the sum of the altitude and a side.



SOLUTION. Let b be the required base and let $a + s$ be the sum of the altitude and a side. Imagine the problem solved, giving $\triangle ABC$. By marking the given lines and studying the figure, the following procedure will be found to be possible: On an unlimited line CE lay off $CB = b$; next draw the $\perp$ bisector of CB and upon it lay off $FD = a + s$. It is now necessary to *break off* a part of $a + s$ to form AB. The fact that AB must equal AD should suggest an isosceles $\triangle$ of which BD will be the base and of which it is required to find the vertex. But the $\perp$ bisector of the base of an isosceles $\triangle$ *passes through* the vertex. Therefore, the solution is completed by drawing the $\perp$ bisector of BD, which determines A, and by drawing AB and AC.

Ex. 188. Construct a triangle, given the base, an adjacent angle, and the sum of the other two sides.

Ex. 189. Construct a right triangle, given one arm and the sum of the hypotenuse and the other arm.

PARALLEL LINES

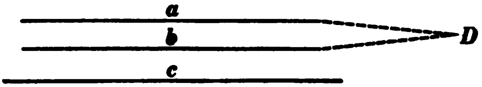
177. Def. Two lines are **parallel** if they lie in the same plane and do not meet however far they are prolonged either way.

178. Assumption 19. Parallel line postulate. *Two intersecting straight lines cannot both be parallel to the same straight line.*

179. The following form of this postulate is sometimes more convenient to quote: *Through a given point there exists only one line parallel to a given line.*

PROPOSITION XXI. THEOREM

180. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



Given lines a and b, each ∥ c.

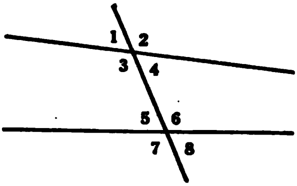
To prove $a \parallel b$.

ARGUMENT	REASONS
1. a and b are either ∥ or not ∥.	1. In this case only two suppositions are admissible.
2. Suppose that a is not ∥ b; then they will meet at some point as D.	2. By def. of ∥ lines. § 177.
3 This is impossible.	3. Parallel line post. § 178.
4. $\therefore a \parallel b$. Q.E.D.	4. The supposition that a and b are not ∥ has been proved false.

181. Def. A **transversal** is a line that intersects two or more other lines.

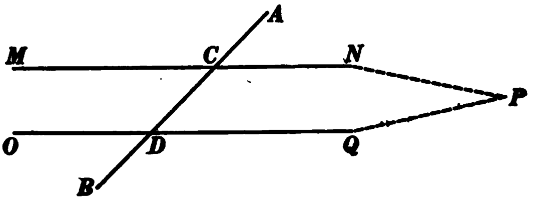
182. Defs. If two straight lines are cut by a transversal, of the eight angles formed,

3, 4, 5, 6 are **interior** angles;
1, 2, 7, 8 are **exterior** angles;
4 and 5, 3 and 6, are **alternate interior** angles;
1 and 8, 2 and 7, are **alternate exterior** angles;
1 and 5, 3 and 7, 2 and 6, 4 and 8, are **corresponding** angles (called also **exterior interior** angles)



Proposition XXII. Theorem

183. *If two straight lines are cut by a transversal making a pair of alternate interior angles equal, the lines are parallel.*



Given two str. lines MN and OQ cut by the transversal AB in points C and D, making $\angle MCD = \angle QDC$.

To prove $MN \parallel OQ$.

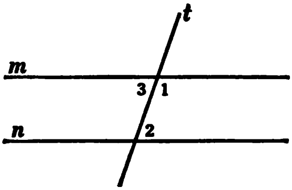
Argument	Reasons
1. MN and OQ are either $\parallel$ or not $\parallel$.	1. In this case only two suppositions are admissible.
2. Suppose that MN is not $\parallel OQ$; then they will meet at some point as P, forming, with line DC, $\triangle PDC$.	2. By def. of $\parallel$ lines. § 177.
3. Then $\angle MCD > \angle QDC$.	3. If one side of a $\triangle$ is prolonged, the ext. $\angle$ formed $>$ either of the remote int. $\angle$s. § 153.
4. This is impossible.	4. $\angle MCD = \angle QDC$, by hyp.
5. ∴ $MN \parallel OQ$.	5. The supposition that MN and OQ are not $\parallel$ has been proved false.
Q.E.D.	

184. Cor. I. *If two straight lines are cut by a transversal making a pair of corresponding angles equal, the lines are parallel.*

Hint. Prove a pair of alt. int. $\angle$s equal, and apply the theorem.

185. **Cor. II.** *If two straight lines are cut by a transversal making a pair of alternate exterior angles equal, the lines are parallel.* (Hint. Prove a pair of alt. int. ∠s equal.)

186. **Cor. III.** *If two straight lines are cut by a transversal making the sum of the two interior angles on the same side of the transversal equal to two right angles, the lines are parallel.*



Given lines m and n cut by the transversal t making

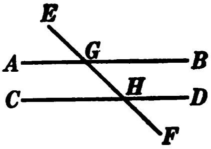
$$\angle 1 + \angle 2 = 2 \text{ rt. } \angle s.$$

To prove $m \parallel n$.

Argument	Reasons
1. $\angle 1 + \angle 2 = 2$ rt. ∠s.	1. By hyp.
2. $\angle 1 + \angle 3 = 2$ rt. ∠s.	2. If one str. line meets another str. line, the sum of the two adj. ∠s is 2 rt. ∠s. § 65.
3. $\therefore \angle 1 + \angle 2 = \angle 1 + \angle 3$.	3. Things equal to the same thing are equal to each other. § 54, 1.
4. $\therefore \angle 2 = \angle 3$.	4. If equals are subtracted from equals, the remainders are equal. § 54, 3.
5. $\therefore m \parallel n$. Q.E.D.	5. If two str. lines are cut by a transversal making a pair of alt. int. ∠s equal, the lines are ∥. § 183.

187. **Cor. IV.** *If two straight lines are perpendicular to a third straight line, they are parallel to each other.*

Ex. 190. If two straight lines are cut by a transversal making the sum of the two exterior angles on the same side of the transversal equal to two right angles, the lines are parallel.



Ex. 191. In the annexed diagram, if angle BGE = angle CHF, are AB and CD parallel? Prove.

Ex. 192. In the same diagram, if angle HGB = angle GHC, prove that the bisectors of these angles are parallel.

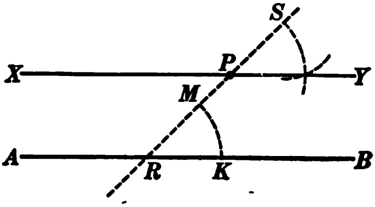
Ex. 193. If two straight lines bisect each other, the lines joining their extremities are parallel in pairs.

Ex. 194. In the diagram for Ex. 191, if angle BGE and angle FHD are supplementary, prove AB parallel to CD.

Ex. 195. If two adjacent angles of any quadrilateral are supplementary, two sides of the quadrilateral will be parallel.

PROPOSITION XXIII. PROBLEM

188. *Through a given point to construct a line parallel to a given line.*



Given line AB and point P.

To construct, through point P, a line ∥ AB.

I. Construction

1. Draw a line through P cutting AB at some point, as R.
2. With P as vertex and PS as side, construct $\angle YPS = \angle BRP$. § 125.
3. XY will be ∥ AB.

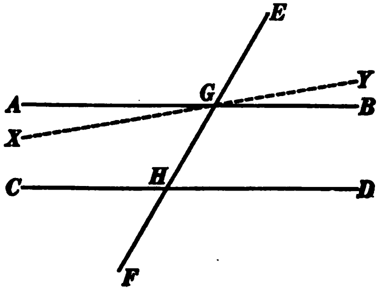
II. The proof and discussion are left as an exercise for the student.

Ex. 196. Through a given point construct a parallel to a given line by using: (*a*) § 183; (*b*) § 185; (*c*) § 187.

Proposition XXIV. Theorem

(Converse of Prop. XXII)

189. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



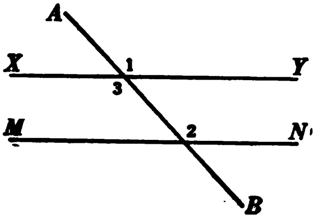
Given $\parallel$ lines AB and CD cut by the transversal EF at points G and H.

To prove $\angle AGH = \angle DHG$.

Argument	Reasons
1. Either $\angle AGH = \angle DHG$, or $\angle AGH \neq \angle DHG$.	1. In this case only two suppositions are admissible.
2. Suppose $\angle AGH \neq \angle DHG$, but that line XY, through G, makes $\angle XGH = \angle DHG$.	2. With a given vertex and a given side, an $\angle$ may be constructed equal to a given $\angle$. § 125.
3. Then $XY \parallel CD$.	3. If two str. lines are cut by a transversal making a pair of alt. int. $\angle$s equal, the lines are $\parallel$. § 183.
4. But $AB \parallel CD$.	4. By hyp.
5. It is impossible that AB and XY both are $\parallel$ CD.	5. Parallel line post. § 178.
6. $\therefore \angle AGH = \angle DHG$.	6. The supposition that $\angle AGH \neq \angle DHG$ has been proved false.

Q.E.D.

190. Cor. I (Converse of Cor. I of Prop. XXII). *If two parallel lines are cut by a transversal, the corresponding angles are equal.*



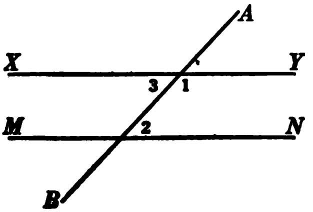
Given two ∥ lines *XY* and *MN* cut by the transversal *AB*, forming corresponding ∠s 1 and 2.

To prove $\angle 1 = \angle 2$.

Hint. $\angle 3 = \angle 2$ by § 189.

191. Cor. II. (Converse of Cor. II of Prop. XXII). *If two parallel lines are cut by a transversal, the alternate exterior angles are equal.*

192. Cor. III. (Converse of Cor. III of Prop. XXII). *If two parallel lines are cut by a transversal, the sum of the two interior angles on the same side of the transversal is two right angles.*



Given two ∥ lines *XY* and *MN* cut by the transversal *AB*, forming int. ∠s 1 and 2.

To prove $\angle 1 + \angle 2 = 2$ rt. ∠s.

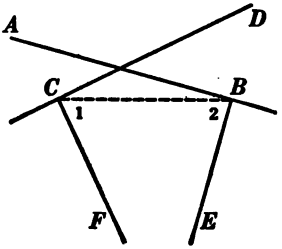
Argument Only

1. $\angle 1 + \angle 3 = 2$ rt. ∠s.
2. $\angle 3 = \angle 2$.
3. $\therefore \angle 1 + \angle 2 = 2$ rt. ∠s. Q.E.D.

193. Cor. IV. *A straight line perpendicular to one of two parallels is perpendicular to the other also.*

194. Cor. V. (Opposite of Cor. III of Prop. XXII). *If two straight lines are cut by a transversal making the sum of the two interior angles on the same side of the transversal not equal to two right angles, the lines are not parallel.* (Hint. Apply § 187, or use the indirect method.)

195. Cor. VI. *Two lines perpendicular respectively to two intersecting lines also intersect.*



Given two intersecting lines AB and CD, and $BE \perp AB$, $CF \perp CD$.

To prove that BE and CF also intersect.

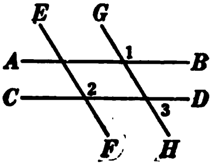
Argument Only

1. Draw CB.
2. $\angle FCD$ is a rt. $\angle$.
3. $\therefore \angle 1 <$ a rt. $\angle$.
4. Likewise $\angle 2 <$ a rt. $\angle$.
5. $\therefore \angle 1 + \angle 2 < 2$ rt. $\angle$s.
6. $\therefore$ BE and CF also intersect. Q.E.D.

196. Def. Two or more lines are said to be **concurrent** if they intersect at a common point.

Ex. 197. If two parallels are cut by a transversal so that one of the angles formed is 45°, how many degrees are there in each of the other seven angles?

Ex. 198. If a quadrilateral has two of its sides parallel, two pairs of its angles will be supplementary.



Ex. 199. In the annexed diagram AB is parallel to CD, and EF is parallel to GH. Prove

(*a*) angle 1 = angle 2;

(*b*) angle 1 + angle 3 = 2 right angles.

Ex. 200. If a line is drawn through any point in the bisector of an angle, parallel to one of the sides of the angle, an isosceles triangle will be formed.

Ex. 201. Draw a line parallel to the base of a triangle, cutting the sides so that the sum of the segments adjacent to the base shall equal the parallel line.

Ex. 202. If two parallel lines are cut by a transversal, the bisectors of a pair of corresponding angles are parallel.

Ex. 203. State and prove the converse of Ex. 190.

Ex. 204. State and prove the converse of Ex. 202.

Ex. 205. If a line is drawn through the vertex of an isosceles triangle parallel to the base, this line bisects the exterior angle at the vertex.

Ex. 206. If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.

Ex. 207. If a line joining two parallels is bisected, any other line through the point of bisection and limited by the parallels is bisected.

Ex. 208. If through the vertex of an acute angle of a right triangle a line is drawn parallel to the opposite side, the line forms with the hypotenuse an angle equal to the other acute angle of the triangle.

Ex. 209. The acute angles of a right triangle are complementary.

Ex. 210. If two sides of a triangle are prolonged their own lengths through the common vertex, the line joining their ends is parallel to the third side of the triangle.

Ex. 211. In an isosceles triangle, if equal segments measured from the vertex are laid off on the arms, the line joining the ends of the segments is parallel to the base of the triangle. (Hint. Draw the bisector of the vertex ∠ of the △.)

Ex. 212. Extend Ex. 211 to the case where the segments are external to the triangle.

SUMMARY OF THEOREMS FOR PROVING LINES PARALLEL

197. (1) If two straight lines are cut by a transversal making the
- alternate interior angles equal,
- alternate exterior angles equal,
- corresponding angles equal,
- interior angles on one side of the transversal supplementary,

the lines are parallel.

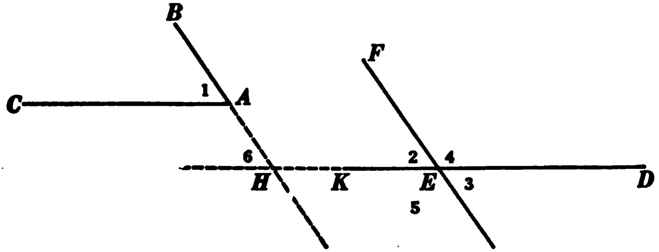
(2) Two straight lines perpendicular to a third straight line are parallel to each other.

(3) Two straight lines parallel to a third straight line are parallel to each other.

(4) After parallelograms have been studied, the fact that the opposite sides of a parallelogram are parallel may be used (§ 220).

PROPOSITION XXV. THEOREM

198. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



Given $\angle 1$ and the $\angle$s at E, with $AB \parallel EF$ and $AC \parallel DE$.

To prove $\angle 1 = \angle 2 = \angle 3$, and $\angle 1 + \angle 4 = 2$ rt. $\angle$s, $\angle 1 + \angle 5 = 2$ rt. $\angle$s.

ARGUMENT	REASONS
1. Prolong BA and DE until they intersect at some point as H.	1. Str. line post. II. § 54, 16.
2. $\angle 1 = \angle 6$.	2. Corresponding $\angle$s of $\parallel$ lines are equal. § 190.
3. $\angle 2 = \angle 6$.	3. Same reason as 2.
4. $\therefore \angle 1 = \angle 2$.	4. Things equal to the same thing are equal to each other. § 54, 1.
5. $\angle 3 = \angle 2$.	5. If two str. lines intersect, the vertical $\angle$s are equal. § 77.
6. $\therefore \angle 1 = \angle 3$.	6. Same reason as 4.
7. $\angle 2 + \angle 4 = 2$ rt. $\angle$s.	7. If one str. line meets an other str. line, the sum of the two adj. $\angle$s is 2 rt. $\angle$s. § 65.
8. $\therefore \angle 1 + \angle 4 = 2$ rt. $\angle$s.	8. Substituting $\angle 1$ for its equal $\angle 2$.

ARGUMENT	REASONS
9. $\angle 5 = \angle 4$.	9. Same reason as 5.
10. $\therefore \angle 1 + \angle 5 = 2$ rt. $\angle$s. Q.E.D.	10. Substituting $\angle 5$ for its equal $\angle 4$.

199. Note. Every angle viewed from its vertex has a right and a left side; thus, in the annexed diagram, the right side of $\angle A$ is r and the left side, l; the right side of $\angle B$ is r and the left, l.

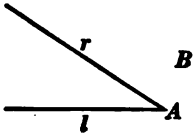


FIG. 1.

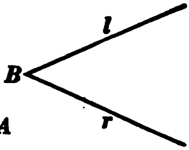


FIG. 2.

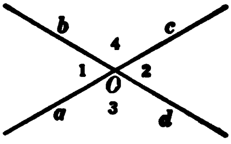


FIG. 3.

In the diagram of Prop. XXV, $\angle 1$ and $\angle 2$, whose sides are $\parallel$ right to right (AB to EF) and left to left (AC to EK) are equal; while $\angle 1$ and $\angle 4$, whose sides are $\parallel$ right to left (AB to EF) and left to right (AC to ED) are supplementary. Hence:

200. (*a*) *If two angles have their sides parallel right to right and left to left, they are equal.*

(*b*) *If two angles have their sides parallel right to left and left to right, they are supplementary.*

Ex. 213. In Fig. 3, above, show which is the right side of each of the four angles about O.

Ex. 214. If a quadrilateral has its opposite sides parallel, its opposite angles are equal.

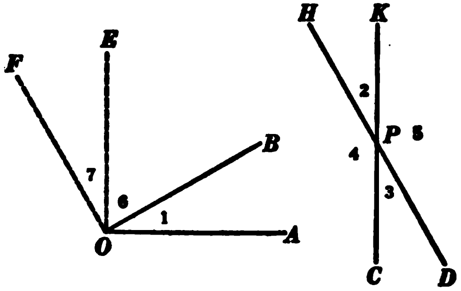
Ex. 215. Given two equal angles having a side of one parallel to a side of the other, are the other sides necessarily parallel? Prove.

Ex. 216. If two sides of a triangle are parallel respectively to two homologous sides of an equal triangle, right to right and left to left, the third side of the first is parallel to the third side of the second.

Ex. 217. Construct a triangle, given an angle and its bisector and the altitude drawn from the vertex of the given angle.

PROPOSITION XXVI. THEOREM

201. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Given $\angle 1$ and the $\angle s$ at P, with $OA \perp KC$ and $OB \perp HD$.

To prove $\angle 1 = \angle 2 = \angle 3$, $\angle 1 + \angle 4 = 2$ rt. $\angle s$, $\angle 1 + \angle 5 = 2$ rt. $\angle s$.

HINT. Draw $OE \parallel CK$ and $OF \parallel DH$. Prove $OE \perp OA$ and $OF \perp OB$. Prove $\angle 7 = \angle 1$, and prove $\angle 7 = \angle 2$.

202. Note. It will be seen that $\angle 1$ and $\angle 2$, whose sides are $\perp$ right to right (OA to PK) and left to left (OB to PH), are equal; while $\angle 1$ and $\angle 4$, whose sides are $\perp$ right to left (OA to PC) and left to right (OB to PH), are supplementary. Hence:

203. (*a*) *If two angles have their sides perpendicular right to right and left to left, they are equal.*

(*b*) *If two angles have their sides perpendicular right to left and left to right, they are supplementary.*

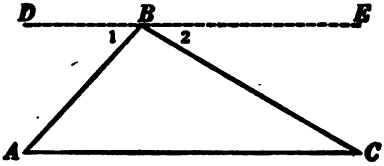
Ex. 218. If from a point outside of an angle perpendiculars are drawn to the sides of the angle, an angle is formed which is equal to the given angle.

Ex. 219. In a right triangle if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the right angle is divided into two angles which are equal respectively to the acute angles of the triangle.

Ex. 220. If from the end of the bisector of the vertex angle of an isosceles triangle a perpendicular is dropped upon one of the arms, the perpendicular forms with the base an angle equal to half the vertex angle.

PROPOSITION XXVII. THEOREM

204. *The sum of the angles of any triangle is two right angles.*



Given $\triangle ABC$.

To prove $\angle A + \angle ABC + \angle C = 2$ rt. $\angle$s.

ARGUMENT	REASONS
1. Through B draw $DE \parallel AC$.	1. Parallel line post. § 179.
2. $\angle 1 + \angle ABC + \angle 2 = 2$ rt. $\angle$s.	2. The sum of all the $\angle$s about a point on one side of a str. line passing through that point = 2 rt. $\angle$s. § 66.
3. $\angle 1 = \angle A$.	3. Alt. int. $\angle$s of $\parallel$ lines are equal. § 189.
4. $\angle 2 = \angle C$.	4. Same reason as 3.
5. $\therefore \angle A + \angle ABC + \angle C = 2$ rt. $\angle$s. Q.E.D.	5. Substituting for $\angle$s 1 and 2 their equals, $\angle$s A and C, respectively.

205. Cor. I. *In a right triangle the two acute angles are complementary.*

206. Cor. II. *In a triangle there can be but one right angle or one obtuse angle.*

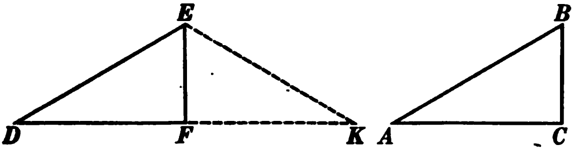
207. Cor. III. *If two angles of one triangle are equal respectively to two angles of another, then the third angle of the first is equal to the third angle of the second.*

208. Cor. IV. *If two right triangles have an acute angle of one equal to an acute angle of the other, the other acute angles are equal.*

209. **Cor. V.** *Two right triangles are equal if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.*

210. **Cor. VI.** *Two right triangles are equal if a side and an acute angle of one are equal respectively to a side and the homologous acute angle of the other.*

211. **Cor. VII.** *Two right triangles are equal if the hypotenuse and a side of one are equal respectively to the hypotenuse and a side of the other.*



Given rt. $\triangle$s ABC and DEF, with $AB = DE$ and $BC = EF$.

To prove $\triangle ABC = \triangle DEF$.

Hint. Prove DFK a str. line; then $\triangle DEK$ is isosceles.

212. **Cor. VIII.** *The altitude upon the base of an isosceles triangle bisects the base and also the vertex angle.*

213. **Cor. IX.** *Each angle of an equilateral triangle is one third of two right angles, or 60°.*

214. **Question.** Why is the word *homologous* used in Cor. VI but not in Cor. V?

Ex. 221. If any angle of an isosceles triangle is 60°, what is the value of each of the two remaining angles?

Ex. 222. If the vertex angle of an isosceles triangle is 20°, find the angle included by the bisectors of the base angles.

Ex. 223. Find each angle of a triangle if the second angle equals twice the first and the third equals three times the second.

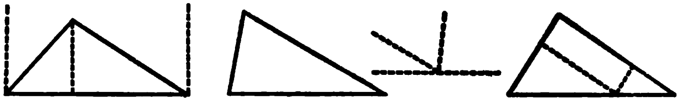
Ex. 224. If one angle of a triangle is $m°$ and another angle $l°$, write an expression for the third angle.

Ex. 225. If the vertex angle of an isosceles triangle is $a°$, write an expression for each base angle.

Ex. 226. Construct an angle of 60°; 120°; 30°; 15°.

Ex. 227 Construct a right triangle having one of its acute angles 60°. How large is the other acute angle?

Ex. 228. Construct an angle of 150°



Ex. 229. Prove Prop. XXVII by using each of the diagrams given above.

Ex. 230 Given two angles of a triangle, construct the third angle.

Ex. 231. Find the sum of the angles of a quadrilateral.

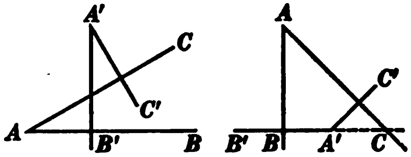
Ex. 232. The angle between the bisectors of two adjacent angles of a quadrilateral is equal to half the sum of the two remaining angles.

Ex. 233. The angle between the bisectors of the base angles of an isosceles triangle is equal to the exterior angle formed by prolonging the base.

Ex. 234. If two straight lines are cut by a transversal and the bisectors of two interior angles on the same side of the transversal are perpendicular to each other, the lines are parallel.

Ex. 235. If in an isosceles triangle each of the base angles is one fourth the angle at the vertex, a line drawn perpendicular to the base at one of its ends and meeting the opposite side prolonged will form with the adjacent side and the exterior portion of the opposite side an equilateral triangle.

Ex. 236. Two angles whose sides are perpendicular each to each are either equal or supplementary. (Prove by using the annexed diagram.)

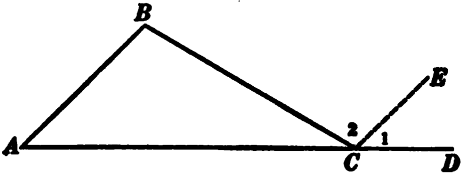


Ex. 237. If at the ends of the hypotenuse of a right triangle perpendiculars to the hypotenuse are drawn meeting the other two sides of the triangle prolonged, then the figure contains five triangles which are mutually equiangular.

Ex. 238. If one angle of a triangle is 50° and another angle is 70°, find the other interior angle of the triangle; also the exterior angles of the triangle. What relation is there between an exterior angle and the two remote interior angles of the triangle?

PROPOSITION XXVIII. THEOREM

215. *An exterior angle of a triangle is equal to the sum of the two remote interior angles.*



Given $\triangle ABC$ with $\angle DCB$ an exterior $\angle$.

To prove $\angle DCB = \angle A + \angle B$.

The proof is left as an exercise for the student.

HINT. Draw $CE \parallel AB$.

Ex. 239. The bisector of an exterior angle at the vertex of an isosceles triangle is parallel to the base.

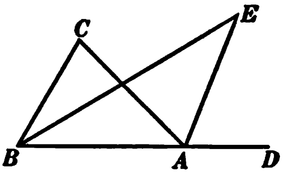
Ex. 240. If the sum of two exterior angles of a triangle is equal to three right angles, the triangle is a right triangle.

Ex. 241. The sum of the three exterior angles of a triangle is four right angles.

Ex. 242. What is the sum of the exterior angles of a quadrilateral?

Ex. 243. If the two exterior angles at the base of any triangle are bisected, the angle between these bisectors is equal to half the sum of the interior base angles of the triangle.

Ex. 244. If BE bisects angle B of triangle ABC, and AE bisects the exterior angle DAC, angle E is equal to one half angle C.

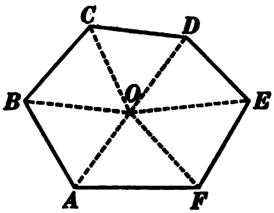


Ex. 245. D is any point in the base BC of isosceles triangle ABC. The side AC is prolonged through C to E so that $CE = CD$, and DE is drawn meeting AB at F. Prove angle EFA equal to three times angle AEF.

Ex. 246. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices. Prove by laying off on the right angle either acute angle.

PROPOSITION XXIX. THEOREM

216. *The sum of all the angles of any polygon is twice as many right angles as the polygon has sides, less four right angles.*



Given polygon $ABCDE\cdots$, any polygon having n sides.

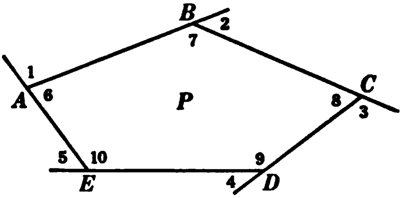
To prove the sum of its $\angle$s $= 2n$ rt. $\angle$s $-$ 4 rt. $\angle$s.

ARGUMENT	REASONS
1. From any point within the polygon such as O, draw lines to the vertices.	1. Straight line post I. § 54, 15.
2. There will be formed n △s.	2. Each side of the polygon will become the base of a △.
3. The sum of the $\angle$s of each △ thus formed = 2 rt. $\angle$s.	3. The sum of the $\angle$s of any △ is 2 rt. $\angle$s. § 204.
4. ∴ the sum of the $\angle$s of the n △s thus formed $= 2n$ rt. $\angle$s.	4. If equals are multiplied by equals, the products are equal. § 54, 7 *a*.
5. The sum of all the $\angle$s about O = 4 rt. $\angle$s.	5. The sum of all the $\angle$s about a point = 4 rt. $\angle$s. § 67.
6. ∴ the sum of all the $\angle$s of the polygon $= 2n$ rt. $\angle$s $-$ 4 rt. $\angle$s. Q.E.D.	6. The sum of all the $\angle$s of the polygon = the sum of the $\angle$s of the n △s $-$ the sum of all the $\angle$s about O.

217. Cor. *Each angle of an equiangular polygon of n sides is equal to* $\frac{2(n-2)}{n}$ *right angles.*

Proposition XXX. Theorem

218. *If the sides of any polygon are prolonged in succession one way, no two adjacent sides being prolonged through the same vertex, the sum of the exterior angles thus formed is four right angles.*



Given polygon P with $\angle 1, \angle 2, \angle 3, \angle 4, \cdots$ its successive exterior angles.

To prove $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \cdots = 4$ rt. $\angle$s.

Argument	Reasons
1. $\angle 1 + \angle 6 = 2$ rt. $\angle$s, $\angle 2 + \angle 7 = 2$ rt. $\angle$s, and so on; *i.e.* the sum of the int. $\angle$ and the ext. $\angle$ at one vertex $= 2$ rt. $\angle$s.	1. If one str. line meets another str. line, the sum of the two adj. $\angle$s is 2 rt. $\angle$s. § 65.
2. $\therefore$ the sum of the int. and ext. $\angle$s at the n vertices $= 2n$ rt. $\angle$s.	2. If equals are multiplied by equals, the products are equal. § 54, 7 *a*.
3. Denote the sum of all the interior $\angle$s by I and the sum of all the ext. $\angle$s by E; then $E + I = 2n$ rt. $\angle$s.	3. Arg. 2.
4. But $I = 2n$ rt. $\angle$s $-$ 4 rt. $\angle$s.	4. The sum of all the $\angle$s of any polygon $= 2n$ rt. $\angle$s $-$ 4 rt. $\angle$s. § 216.
5. $\therefore E = 4$ rt. $\angle$s; *i.e.* $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \cdots = 4$ rt. $\angle$s. Q.E.D.	5. If equals are subtracted from equals, the remainders are equal. § 54, 3

219. Note. The formula $2n$ rt. $\angle$s $- 4$ rt. $\angle$s (§ 216) is sometimes more useful in the form $(n - 2)\ 2$ rt. $\angle$s.

Ex. 247. Find the sum of the angles of a polygon of 7 sides; of 8 sides; of 10 sides.

Ex. 248. Prove Prop. XXIX by drawing as many diagonals as possible from one vertex.

Ex. 249. How many diagonals can be drawn from one vertex in a polygon of 8 sides? of 50 sides? of n sides? Show that the greatest number of diagonals possible in a polygon of n sides (using all vertices) is $\frac{n(n-3)}{2}$.

Ex. 250. How many degrees are there in each angle of an equiangular quadrilateral? in each angle of an equiangular pentagon?

Ex. 251. How many sides has a polygon the sum of whose angles is 14 right angles? 20 right angles? 540°?

Ex. 252. How many sides has a polygon the sum of whose interior angles is double the sum of its exterior angles?

Ex. 253. Is it possible for an exterior angle of an equiangular polygon to be 70°? 72°? 140°? 144°?

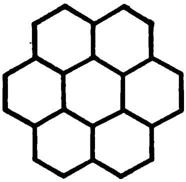
Ex. 254. How many sides has a polygon each of whose exterior angles equals 12°?

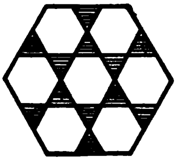
Ex. 255. How many sides has a polygon each of whose exterior angles is one eleventh of its adjacent interior angle?

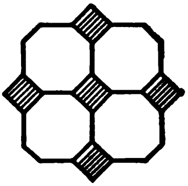
Ex. 256. How many sides has a polygon the sum of whose interior angles is six times the sum of its exterior angles?

Ex. 257. How many sides has an equiangular polygon if the sum of three of its exterior angles is 180°?

Ex. 258. Tell what equiangular polygons can be put together to make a pavement. How many equiangular triangles must be placed with a common vertex to fill the angular magnitude around a point?







QUADRILATERALS. PARALLELOGRAMS

QUADRILATERALS CLASSIFIED WITH RESPECT TO PARALLELISM

220. Def. A **parallelogram** is a quadrilateral whose opposite sides are parallel.

221. Def. A **trapezoid** is a quadrilateral having two of its opposite sides parallel and the other two not parallel.

222. Def. A **trapezium** is a quadrilateral having no two of its sides parallel.

PARALLELOGRAMS CLASSIFIED WITH RESPECT TO ANGLES

223. Def. A **rectangle** is a parallelogram having one right angle.

It is shown later that all the angles of a rectangle are right angles.

224. Def. A **rhomboid** is a parallelogram having an oblique angle.

It is shown later that all the angles of a rhomboid are oblique.

225. Def. A rectangle having two adjacent sides equal is a **square.**

It is shown later that all the sides of a square are equal.

226. Def. A rhomboid having two adjacent sides equal is a **rhombus.**

It is shown later that all the sides of a rhombus are equal.

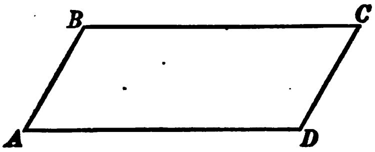
227. Def. A trapezoid having its two non-parallel sides equal is an **isosceles trapezoid.**

228. Def. Any side of a parallelogram may be regarded as its **base**, and the line drawn perpendicular to the base from any point in the opposite side is then the **altitude.**

229. Def. The **bases** of a trapezoid are its parallel sides, and its **altitude** is a line drawn from any point in one base perpendicular to the other.

PROPOSITION XXXI. THEOREM

230. *Any two opposite angles of a parallelogram are equal; and any two consecutive angles are supplementary.*



Given ▱ $ABCD$.

To prove: (*a*) $\angle A = \angle C$, and $\angle B = \angle D$;
(*b*) any two consecutive $\angle$s, as A and B, sup.

ARGUMENT	REASONS
1 $\angle A = \angle C$ and $\angle B = \angle D$.	1. If two $\angle$s have their sides ∥ right to right and left to left, they are equal. § 200, *a*.
2. $\angle$s A and B are sup. Q.E.D.	2. If two ∥ lines are cut by a transversal, the sum of the two int. $\angle$s on the same side of the transversal is two rt. $\angle$s. § 192.

231. Cor. ***All the angles of a rectangle are right angles, and all the angles of a rhomboid are oblique angles.***

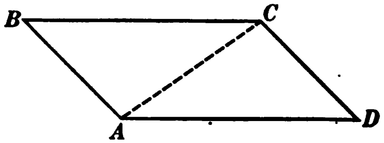
Ex. 259. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

Ex. 260. If an angle of one parallelogram is equal to an angle of another, the remaining angles are equal each to each.

Ex. 261. The bisectors of the angles of a parallelogram (not a rhombus or a square) inclose a rectangle.

PROPOSITION XXXII. THEOREM

232. *The opposite sides of a parallelogram are equal.*



Given ▱ $ABCD$.

To prove $AB = CD$ and $BC = AD$.

The proof is left as an exercise for the student.

233. **Cor. I.** *All the sides of a square are equal, and all the sides of a rhombus are equal.*

234. **Cor. II.** *Parallel lines intercepted between the same parallel lines are equal.*

235. **Cor. III.** *The perpendiculars drawn to one of two parallel lines from any two points in the other are equal.*

236. **Cor. IV.** *A diagonal of a parallelogram divides it into two equal triangles.*

Ex. 262. The perpendiculars drawn to a diagonal of a parallelogram from the opposite vertices are equal.

Ex. 263. The diagonals of a rhombus are perpendicular to each other and so are the diagonals of a square.

Ex. 264. The diagonals of a rectangle are equal.

Ex. 265. The diagonals of a rhomboid are unequal.

Ex. 266. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Ex. 267. If the diagonals of a parallelogram are not equal, the figure is a rhomboid.

Ex. 268. Draw a line parallel to the base of a triangle so that the portion intercepted between the sides may be equal to a given line.

Ex. 269. Explain the statement: Parallel lines are everywhere equidistant. Has this been proved?

Ex. 270. Find the locus of a point that is equidistant from two given parallel lines.

Ex. 271. Find the locus of a point: (*a*) one inch above a given horizontal line; (*b*) two inches below the given line.

Ex. 272. Find the locus of a point: (*a*) one inch to the right of a given vertical line; (*b*) one inch to the left of the given line.

Ex. 273. Given a horizontal line OX and a line OY perpendicular to OX. Find the locus of a point three inches above OX and two inches to the right of OY.

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237. Historical Note. René Descartes (1596–1650) was the first to observe the importance of the fact that the position of a point in a plane is determined if its distances, say x and y, from two fixed lines in the plane, perpendicular to each other, are known. He showed that geometric figures can be represented by algebraic equations, and developed the subject of analytic geometry, which is known by his name as *Cartesian* geometry.

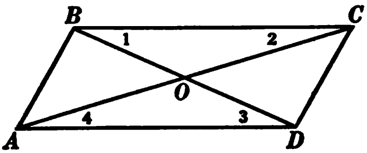
DESCARTES

Descartes was born near Tours in France, and was sent at eight years of age to the famous Jesuit school at La Flêche. He was of good family, and since, at that time, most men of position entered either the church or the army, he chose the latter, and joined the army of the Prince of Orange.

One day, while walking in a street in a Holland town, he saw a placard which challenged every one who read it to solve a certain geometric problem. Descartes solved it with little difficulty and is said to have realized then that he had no taste for military life. He soon resigned his commission and spent five years in travel and study. After this he lived a short time in Paris, but soon retired to Holland, where he lived for twenty years, devoting his time to mathematics, philosophy, astronomy, and physics. His work in philosophy was of such importance as to give him the name of the Father of Modern Philosophy

Proposition XXXIII. Theorem

238. *The diagonals of a parallelogram bisect each other.*



Given ▱ $ABCD$ with its diagonals AC and BD intersecting at O.

To prove $AO = OC$ and $BO = OD$.

Hint. Prove $\triangle OBC = \triangle ODA$. $\therefore AO = OC$ and $BO = OD$.

Ex. 274. If through the vertices of a triangle lines are drawn parallel to the opposite sides of the triangle, the lines which join the vertices of the triangle thus formed to the opposite vertices of the given triangle are bisected by the sides of the given triangle.

Ex. 275. A line terminated by the sides of a parallelogram and passing through the point of intersection of its diagonals is bisected at that point.

Ex. 276. How many parallelograms can be constructed having a given base and altitude? What is the locus of the point of intersection of the diagonals of all these parallelograms?

Ex. 277. If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus or a square.

Ex. 278. If an angle of a parallelogram is bisected by one of the diagonals, the figure is a rhombus or a square.

Ex. 279. Find on one side of a triangle the point from which straight lines drawn parallel to the other two sides, and terminated by those sides, are equal. (See § 232.)

Ex. 280. Find the locus of a point at a given distance from a given finite line AB.

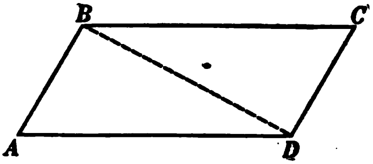
Ex. 281. Find the locus of a point at a given distance from a given line and also equidistant from the ends of another given line.

Ex. 282. Construct a parallelogram, given a side, a diagonal, and the altitude upon the given side.

PROPOSITION XXXIV. THEOREM

(Converse of Prop. XXXII)

239. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given quadrilateral $ABCD$, with $AB = CD$, and $BC = AD$.

To prove $ABCD$ a ▱.

ARGUMENT	REASONS
1. Draw the diagonal BD.	1. Str. line post. I. § 54, 15.
2. In $\triangle$s ABD and BCD, $AB = CD$.	2. By hyp.
3. $BC = AD$.	3. By hyp.
4. $BD = BD$.	4. By iden.
5. $\therefore \triangle ABD = \triangle BCD$.	5. Two $\triangle$s are equal if the three sides of one are equal respectively to the three sides of the other. § 116.
6. $\therefore \angle ABD = \angle CDB$.	6. Homol. parts of equal figures are equal. § 110.
7. $\therefore AB \parallel CD$.	7. If two str. lines are cut by a transversal making a pair of alt. int. ∠s equal, the lines are ∥. § 183.
8. Likewise $\angle BDA = \angle DBC$.	8. Same reason as 6.
9. $\therefore BC \parallel AD$.	9. Same reason as 7.
10. $\therefore ABCD$ is a ▱. Q.E.D.	10. By def. of a ▱. § 220.

Ex. 283. Construct a parallelogram, given two adjacent sides and the included angle.

Ex. 284. Construct a rectangle, given the base and the altitude.

Ex. 285. Construct a square, given a side.

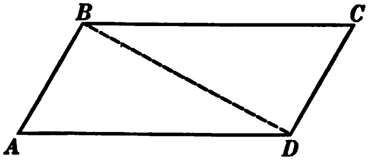
Ex. 286. Through a given point construct a parallel to a given line by means of Prop. XXXIV.

Ex. 287. Construct a median of a triangle by means of a parallelogram, (1) using §§ 239 and 238; (2) using §§ 220 and 238.

Ex. 288. An angle of a triangle is right, acute, or obtuse according as the median drawn from its vertex is equal to, greater than, or less than half the side it bisects.

PROPOSITION XXXV. THEOREM

240. *If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given quadrilateral $ABCD$, with BC both equal and $\parallel$ to AD.

To prove $ABCD$ a ▱.

OUTLINE OF PROOF

1. Draw diagonal BD.
2. Prove $\triangle BCD = \triangle ABD$.
3. Then $\angle CDB = \angle ABD$ and $AB \parallel CD$.
4. $\therefore ABCD$ is a ▱.

Ex. 289. If the mid-points of two opposite sides of a parallelogram are joined to a pair of opposite vertices, a parallelogram will be formed.

Ex. 290. Construct a parallelogram, having given a base, an adjacent angle, and the altitude, making your construction depend upon § 240.

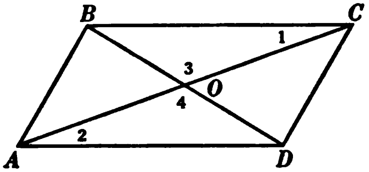
Ex. 291. If the perpendiculars to a line from any two points in another line are equal, then the lines are parallel.

Ex. 292. If two parallelograms have two vertices and a diagonal in common, the lines joining the other four vertices form a parallelogram.

PROPOSITION XXXVI. THEOREM

(Converse of Prop. XXXIII)

241. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Given quadrilateral *ABCD* with its diagonals *AC* and *BD* intersecting at *O* so that $AO = CO$ and $BO = DO$.

To prove *ABCD* a ▱.

ARGUMENT ONLY

1. In ⟁ *OBC* and *ODA*,
   $BO = DO$.
2. $CO = AO$.
3. $\angle 3 = \angle 4$.
4. ∴ $\triangle OBC = \triangle ODA$.
5. ∴ $BC = AD$.
6. Also $\angle 1 = \angle 2$.
7. ∴ $BC \parallel AD$.
8. ∴ *ABCD* is a ▱. Q.E.D

Ex. 293. In parallelogram *ABCD*, let diagonal *AC* be prolonged through *A* and *C* to *X* and *Y*, respectively, making $AX = CY$. Prove *XBYD* a parallelogram.

Ex. 294. If each half of each diagonal of a parallelogram is bisected, the lines joining the points of bisection form a parallelogram.

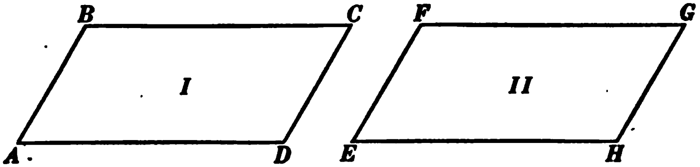
Ex. 295. Lines drawn from the vertices of two angles of a triangle and terminating in the opposite sides cannot bisect each other.

Ex. 296. State four independent hypotheses which would lead to the conclusion, "the quadrilateral is a parallelogram."

Ex. 297. Construct a parallelogram, given its diagonals and an angle between them.

PROPOSITION XXXVII. THEOREM

242. *Two parallelograms are equal if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given ▱s I and II with $AB = EF$, $AD = EH$, and $\angle A = \angle E$.

To prove ▱ I = ▱ II.

ARGUMENT	REASONS
1. Place ▱ I upon ▱ II so that AB shall fall upon its equal EF, A upon E, B upon F.	1. Transference post. § 54, 14
2. Then AD will become collinear with EH.	2. $\angle A = \angle E$, by hyp.
3. Point D will fall on H.	3. $AD = EH$, by hyp.
4. Now $DC \parallel AB$, and $HG \parallel EF$.	4. By def. of a ▱. § 220.
5. $\therefore DC$ and HG are both $\parallel AB$.	5. AB and EF coincide, Arg. 1.
6. $\therefore DC$ will become collinear with HG, and C will fall somewhere on HG or its prolongation.	6. Parallel line post. § 179.
7. Likewise BC will become collinear with FG, and C will fall somewhere on FG or its prolongation.	7. By steps similar to 4, 5, and 6.
8. $\therefore$ point C must fall on point G.	8. Two intersecting str. lines can have only one point in common. § 26.
9. $\therefore$ ▱ I = ▱ II. Q.E.D.	9. By def. of equal figures. § 18.

QUADRILATERALS CLASSIFIED

243. Quadrilaterals

1. Opposite sides ∥: Parallelogram.
    - (*a*) Right-angled: Rectangle.
        - Two adj. sides equal: Square.
    - (*b*) Oblique-angled: Rhomboid.
        - Two adj. sides equal: Rhombus.
2. *Two* sides ∥, other two non-∥: Trapezoid.
    - (*a*) Two non-∥ sides equal: Isosceles trapezoid.
3. *No two* sides ∥: Trapezium.

Ex. 298. If it is required to prove a given quadrilateral a rectangle, show by reference to § 243 that the logical steps are to prove first that it is a parallelogram; then that it has one right angle.

Ex. 299. If a given quadrilateral is to be proved a square, show that the only additional step after those in Ex. 298 is to prove two adjacent sides equal.

Ex. 300. If a given quadrilateral is to be proved a rhombus, what are the three steps corresponding to those given in Ex. 298 and Ex. 299?

Ex. 301. Since rectangles and rhomboids are parallelograms, they possess all the general properties of parallelograms. What property differentiates rectangles from rhomboids (1) by definition? (2) by proof? (See Ex. 266 and Ex. 267.)

Ex. 302. (*a*) What two properties that have been *proved* distinguish squares from other rectangles? (See Ex. 277 and Ex. 278.)

(*b*) What two properties that have been proved distinguish rhombuses from other rhomboids? (See Ex. 277 and Ex. 278.)

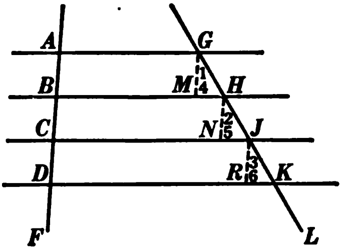
(*c*) Show that the two properties which distinguish squares and rhombuses from the other members of their class are due to the common property possessed by squares and rhombuses by definition.

Ex. 303. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.

Ex. 304. If a line *AB* of given length is moved so that its ends always touch the sides of a given right angle, what is the locus of the mid-point of *AB*?

PROPOSITION XXXVIII. THEOREM

244. *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on any other transversal.*



Given ∥ lines AG, BH, CJ, DK, etc., which intercept the equal segments AB, BC, CD, etc., on transversal AF, and which intercept segments GH, HJ, JK, etc., on transversal GL.

To prove $GH = HJ = JK$, etc.

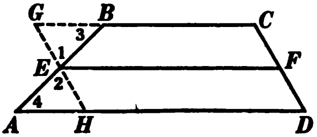
ARGUMENT	REASONS
1. Draw GM, HN, JR, etc. ∥ AF.	1. Parallel line post. § 179.
2. Now $AGMB$, $BHNC$, $CJRD$, etc., are ▱s.	2. By def. of a ▱. § 220.
3. ∴ $GM = AB$, $HN = BC$, $JR = CD$, etc.	3. The opposite sides of a ▱ are equal. § 232.
4. And $AB = BC = CD$, etc.	4. By hyp.
5. ∴ $GM = HN = JR$, etc.	5. Ax. 1. § 54, 1.
6. Again GM, HN, JR, etc., are ∥ to each other.	6. If two str. lines are ∥ to a third str. line, they are ∥ to each other. § 180
7. ∴ $\angle 1 = \angle 2 = \angle 3$, etc.	7. Corresponding ∠s of ∥ lines are equal. § 190.
8. And $\angle 4 = \angle 5 = \angle 6$, etc.	8. If two ∠s have their sides ∥ right to right and left to left, they are equal. § 200, *a*.

9. $\therefore \triangle GHM = \triangle HJN = \triangle JKR$, etc.	9. Two △ are equal if two ∠ and the included side of one are equal respectively to two ∠ and the included side of the other. § 105.
10. $\therefore GH = HJ = JK$, etc. Q.E.D.	10. Homol. parts of equal figures are equal. § 110.

245. Question. Are the segments that the parallels intercept on one transversal equal to the segments that they intercept on another?

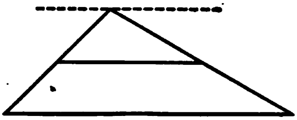
246. Cor. I. *The line bisecting one of the non-parallel sides of a trapezoid and parallel to the bases bisects the other of the non-parallel sides also.*

247. Cor. II. *The line joining the mid-points of the non-parallel sides of a trapezoid is (a) parallel to the bases; and (b) equal to one half their sum.*

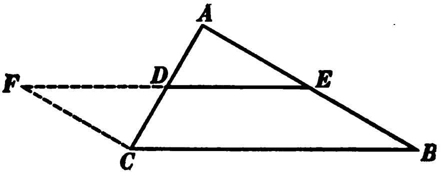


Hint. (*a*) Prove $EF \parallel BC$ and AD by the indirect method. (*b*) Draw $GH \parallel CD$. Prove $AH = GB$; then prove $EF = GC = \frac{1}{2}(GC + HD) = \frac{1}{2}(BC + AD)$.

248. Cor. III. *The line bisecting one side of a triangle and parallel to another side bisects the third side.*



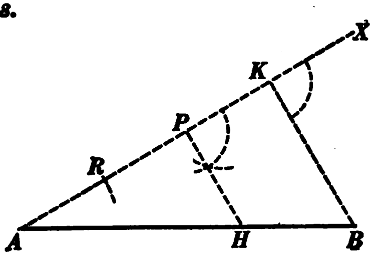
249. Cor. IV. *The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one half the third side.*



Hint. Draw $CF \parallel BA$. Prove $CF = AE = EB$.

PROPOSITION XXXIX. PROBLEM

250. *To divide a given straight line into any number of equal parts.*



Given straight line AB.

To divide AB into n equal parts.

I. Construction

1. Draw the unlimited line AX.

2. Take any convenient segment, as AR, and, beginning at A, lay it off n times on AX.

3. Connect the nth point of division, as K, with B.

4. Through the preceding point of division, as P, draw a line $PH \parallel KB$. § 188.

5. Then HB is one nth of AB.

6. $\therefore$ HB, if laid off successively on AB, will divide AB into n equal parts.

II. The proof and discussion are left as an exercise for the student.

Ex. 305. Divide a straight line into 7 equal parts.

Ex. 306. Construct an equilateral triangle, having given the perimeter.

Ex. 307. Construct a square, having given the perimeter.

251. Def. The line joining the mid-points of the nonparallel sides of a trapezoid is called the **median** of the trapezoid.

Ex. 308. Show, by generalizing, that Cor. III, Prop. XXXVIII, may be obtained from Cor. I and Cor. IV from Cor. II.

Ex. 309. The lines joining the mid-points of the sides of a quadrilateral taken in order form a parallelogram.

Ex. 310. What additional statement can you make if the quadrilateral in Ex. 309 is an isosceles trapezoid? a rectangle? a rhombus? a square?

Ex. 311. Lines drawn from the mid-point of the base of an isosceles triangle to the mid-points of its equal sides form a rhombus or a square. When is the figure a rhombus? when a square?

Ex. 312. The mid-points of the sides of a quadrilateral and the mid-points of its two diagonals are the vertices of three parallelograms whose diagonals are concurrent.

Ex. 313. What is the perimeter of each parallelogram in Ex. 312?

Ex. 314. Construct a triangle, given the mid-points of its sides.

Ex. 315. Through a given point within an angle construct a line, limited by the two sides of the angle, and bisected at the given point.

Ex. 316. Every diagonal of a parallelogram is trisected by the lines joining the other two vertices with the mid-points of the opposite sides.

Ex. 317. If a triangle inscribed in another triangle has its sides parallel respectively to the sides of the latter, its vertices are the mid-points of the sides of the latter.

Ex. 318. If the lower base KT of trapezoid $RSTK$ is double the upper base RS, and the diagonals intersect at O, prove OK double OS, and OT double OR.

Ex. 319. Construct a trapezoid, given the two bases, one diagonal, and one of the non-parallel sides.

In the two following exercises prove the properties which require proof, state those which follow by definition, and those which have been proved in the text:

Ex. 320. *Properties possessed by all trapezoids:*

(*a*) Two sides of a trapezoid are parallel.

(*b*) The two angles adjacent to either of the non-parallel sides are supplementary.

(*c*) The median of a trapezoid is parallel to the bases and equal to one half their sum.

Ex 321. *In an isosceles trapezoid:*

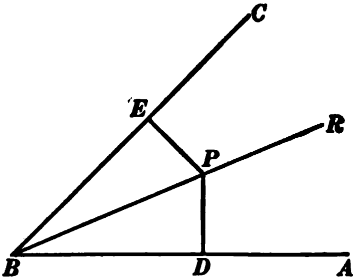
(*a*) The two non-parallel sides are equal.

(*b*) The angles at each base are equal and the opposite angles are supplementary.

(*c*) The diagonals are equal.

Proposition XL. Theorem

252. *The two perpendiculars to the sides of an angle from any point in its bisector are equal.*



Given $\angle ABC$; P any point in BR, the bisector of $\angle ABC$; PD and PE, the ⊥s from P to BA and BC respectively.

To prove $PD = PE$.

Argument Only

1. In rt. ▲s DBP and PBE, $PB = PB$.
2. $\angle DBP = \angle PBE$.
3. $\therefore \triangle DBP = \triangle PBE$.
4. $\therefore PD = PE$. Q.E.D.

253. Prop. XL may be stated as follows:

Every point in the bisector of an angle is equidistant from the sides of the angle.

Ex. 322. Find a point in one side of a triangle which is equidistant from the other two sides of the triangle.

Ex. 323. Find a point equidistant from two given intersecting lines and also at a given distance from a fixed third line.

Ex. 324. Find a point equidistant from two given intersecting lines and also equidistant from two given parallel lines.

Ex. 325. Find a point equidistant from the four sides of a rhombus.

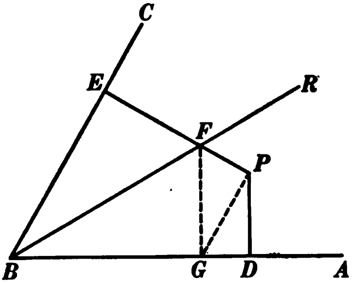
Ex. 326. The two altitudes of a rhombus are equal. Prove.

Ex. 327. Construct the locus of the center of a circle of given radius, which rolls within a given angle so that it always touches a side of the angle. Do not prove.

PROPOSITION XLI. THEOREM

(Opposite of Prop. XL).

254. *The two perpendiculars to the sides of an angle from any point not in its bisector are unequal.*



Given $\angle ABC$; P any point not in BR, the bisector of $\angle ABC$; PD and PE ⊥s from P to BA and BC respectively.

To prove $PD \neq PE$.

OUTLINE OF PROOF

Draw $FG \perp BA$; draw PG. Then $FE = FG$.
Now $PF + FG > PG$. $\therefore PE > PG$. But $PG > PD$.
$\therefore PE > PD$.

255. Prop. XLI may be stated as follows:

Every point not in the bisector of an angle is not equidistant from the sides of the angle.

256. **Cor. I.** (Converse of Prop. XL). *Every point equidistant from the sides of an angle lies in the bisector of the angle.*

HINT. Prove directly, using the figure for § 252, or apply § 137.

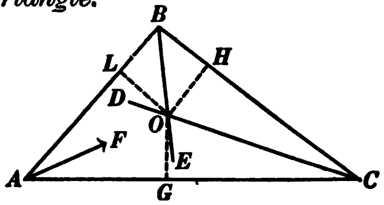
257. **Cor. II.** *The bisector of an angle is the locus of all points equidistant from the sides of the angle.*

Ex. 328. What is the locus of all points that are equidistant from a pair of intersecting lines?

CONCURRENT LINE THEOREMS

PROPOSITION XLII. THEOREM

258. *The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the three sides of the triangle.*



Given $\triangle ABC$ with AF, BE, CD the bisectors of $\angle$s A, B, and C respectively.

To prove: (*a*) AF, BE, CD concurrent in some point as O;
(*b*) the point O equidistant from AB, BC, and CA.

ARGUMENT	REASONS
1. BE and CD will intersect at some point as O.	1. If two str. lines are cut by a transversal making the sum of the two int. $\angle$s on the same side of the transversal not equal to 2 rt. $\angle$s, the lines are not ∥. § 194.
2. Draw OL, OH, and OG, ⊥s from O to AB, BC, and CA respectively.	2. From a point outside a line there exists one and only one ⊥ to the line. § 155.
3. $\because O$ is in BE, $OL = OH$.	3. The two ⊥s to the sides of an $\angle$ from any point in its bisector are equal. § 252.
4. $\because O$ is in CD, $OG = OH$.	4. Same reason as 3.
5. $\therefore OL = OG$.	5. Things equal to the same thing are equal to each other. § 54, 1.

Argument	**Reasons**
6. ∴ AF, the bisector of $\angle CAB$, passes through O.	6. Every point equidistant from the sides of an ∠ lies in the bisector of the ∠. § 256.
7. ∴ AF, BE, and CD are concurrent in O.	7. By def. of concurrent lines. § 196.
8. Also O is equidistant from AB, BC, and CA. Q.E.D.	8. By proof, $OL = OH = OG$.

259. Cor. *The point of intersection of the bisectors of the three angles of a triangle is the locus of all points equidistant from the three sides of the triangle.*

Ex. 329. Is it always possible to find a point equidistant from three given straight lines? from four given straight lines?

Ex. 330. Find a point such that the perpendiculars from it to three sides of a quadrilateral shall be equal. (Give geometric construction.)

Ex. 331. Prove that if the sides AB and AC of a triangle ABC are prolonged to E and F, respectively, the bisectors of the three angles BAC, EBC, and BCF all pass through a point which is equally distant from the three lines AE, AF, and BC. Is any other point in the bisector of the angle BAC equally distant from these three lines? Give reason for your answer.

Ex. 332. Through a given point P draw a straight line such that perpendiculars to it from two fixed points Q and R shall cut off on it equal segments from P. (Hint. See § 246.)

Ex. 333. Construct the locus of the center of a circle of given radius, which rolls so that it always remains inside of a given triangle and constantly touches a side. Do not prove.

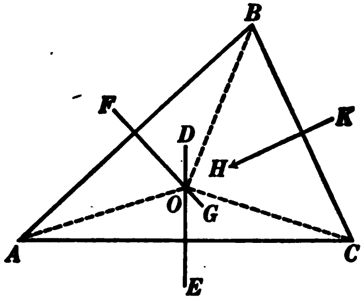
Ex. 334. Find the locus of a point in one side of a parallelogram and equidistant from two other sides. In what parallelograms is this locus a vertex of the parallelogram?

Ex. 335. Find the locus of a point in one side of a parallelogram and equidistant from two of the vertices of the parallelogram. In what class of parallelograms is this locus a vertex of the parallelogram?

Ex. 336. Construct the locus of the center of a circle of given radius which rolls so that it constantly touches a given circumference. Do not prove.

PROPOSITION XLIII. THEOREM

260. *The perpendicular bisectors of the sides of a triangle are concurrent in a point which is equidistant from the three vertices of the triangle.*



Given $\triangle ABC$ with FG, HK, ED, the ⊥ bisectors of AB, BC, CA.

To prove: (*a*) FG, HK, ED concurrent in some point as O;
(*b*) the point O equidistant from A, B, and C.

ARGUMENT	REASONS
1. FG and ED will intersect at some point as O.	1. Two lines ⊥ respectively to two intersecting lines also intersect. § 195.
2. Draw OA, OB, and OC.	2. Str. line post. I. § 54, 15.
3. ∵ O is in FG, the ⊥ bisector of AB, $OB = OA$; and ∵ O is in DE, the ⊥ bisector of CA, $OC = OA$.	3. Every point in the ⊥ bisector of a line is equidistant from the ends of that line. § 134.
4. ∴ $OB = OC$.	4 Ax. 1. § 54, 1.
5. ∴ HK, the ⊥ bisector of BC, passes through O.	5. Every point equidistant from the ends of a line lies in the ⊥ bisector of that line. § 139.
6. ∴ FG, HK, and ED are concurrent in O.	6. By def. of concurrent lines § 196.
7. Also O is equidistant from A, B, and C. Q.E.D.	7. By proof, $OA = OB = OC$.

261. Cor. *The point of intersection of the perpendicular bisectors of the three sides of a triangle is the locus of all points equidistant from the three vertices of the triangle.*

Ex. 337. Is it always possible to find a point equidistant from three given points? from four given points?

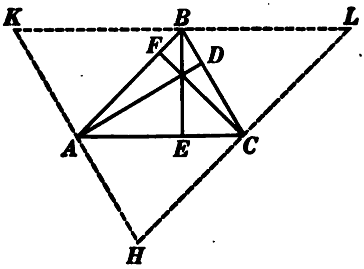
Ex. 338. Construct the perpendicular bisectors of two sides of an acute triangle, and then construct a circle whose circumference shall pass through the vertices of the triangle.

Ex. 339. Construct a circle whose circumference shall pass through the vertices of a right triangle.

Ex. 340. Construct a circle whose circumference shall pass through the vertices of an obtuse triangle.

Proposition XLIV. Theorem

262. *The altitudes of a triangle are concurrent.*



Given $\triangle ABC$ with its altitudes AD, BE, and CF.

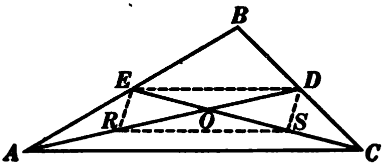
To prove AD, BE, and CF concurrent.

Outline of Proof

Through the vertices A, B, and C, of triangle ABC, draw lines ∥ BC, AC, and AB, respectively. Then prove, by means of ▱s, that AD, BE, and CF are the ⊥ bisectors, respectively, of the sides of the auxiliary $\triangle HKL$. Then, by Prop. XLIII, AD, BE, and CF are concurrent. Q.E.D.

PROPOSITION XLV. THEOREM

263. *Any two medians of a triangle intersect each other in a trisection point of each.*



Given $\triangle ABC$ with AD and CE any two of its medians.

To prove that AD and CE intersect in a point O such that $OD = \frac{1}{3} AD$ and $OE = \frac{1}{3} CE$.

OUTLINE OF PROOF

1. AD and CE will intersect at some point as O. § 194.
2. Let R and S be the mid-points of AO and CO respectively.
3. Quadrilateral $REDS$ is a ▱.
4. $\therefore AR = RO = OD$ and $CS = SO = OE$.
5. That is, $OD = \frac{1}{3} AD$ and $OE = \frac{1}{3} CE$. Q.E.D.

264. Cor. *The three medians of a triangle are concurrent.*

265. Def. The point of intersection of the medians of a triangle is called the **median center** of the triangle. It is also called the **centroid** of the triangle. This point is the center of mass or center of gravity of the triangle.

Ex. 341. Draw a triangle whose altitudes will intersect on one of its sides, and repeat the proof for Prop. XLIV.

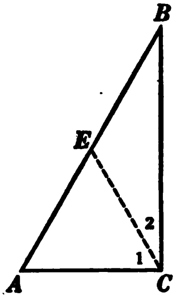
Ex. 342. Draw a triangle whose altitudes will intersect outside of the triangle, and repeat the proof for Prop. XLIV.

Ex. 343. Prove Prop. XLV by prolonging OD its own length and drawing lines to B and C from the end of the prolongation.

Ex. 344. Construct a triangle, given two of its medians and the angle between them.

PROPOSITION XLVI. THEOREM

266. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side.*



Given rt. △ ABC with $\angle A$ double $\angle B$.

To prove $AB = 2\ AC$.

ARGUMENT ONLY

1. Draw CE, making $\angle 1 = \angle A$.
2. $\angle A = 60°$.
3. $\therefore \angle 1 = 60°$.
4. $\therefore \angle AEC = 60°$.
5. $\therefore AE = AC = EC$.
6. In △ EBC, $\angle 2 = 30°$
7. $\angle B = 30°$.
8. $\therefore EB = EC$.
9. $\therefore EB = AE = AC$.
10. $\therefore AB = 2\ AC$. Q.E.D.

267. Prop. XLVI is sometimes stated: *In a thirty-sixty degree right triangle, the hypotenuse is double the shorter side.*

268. Historical Note. Such a triangle (a 30°–60° right triangle) is spoken of by Plato as "the most beautiful right-angled scalene triangle."

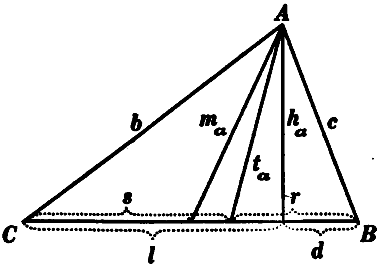
Ex. 345. Prove Prop. XLVI by drawing $CE = CA$.

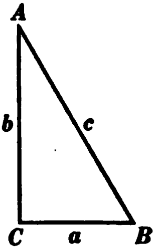
Ex. 346. Prove Prop. XLVI by drawing lines through the ends of the hypotenuse parallel to the other two sides, thus forming a rectangle.

CONSTRUCTION OF TRIANGLES

269. A triangle is *determined*, in general, when *three* parts are given, provided that at least *one* of the given parts is a *line.*

The three sides and the three angles of a triangle are called its *parts;* but there are also many *indirect* parts; as the three medians, the three altitudes, the three bisectors, and the parts into which both the sides and the angles are divided by these lines.





270. The notation given in the annexed figures may be used for brevity:

$A, B, C,$	the angles of the triangle; in a right triangle, angle C is the right angle.
$a, b, c,$	the sides of the triangle; in a right triangle, c is the hypotenuse.
$m_a, m_b, m_c,$	the medians to a, b, and c respectively.
$h_a, h_b, h_c,$	the altitudes to a, b, and c respectively.
$t_a, t_b, t_c,$	the bisectors of A, B, and C respectively.
$l_a, d_a,$	the segments of a made by the altitude to a.
$s_a, r_a,$	the segments of a made by the bisector of angle A.

271. The student should review the chief cases of construction of triangles already given: viz. a, B, c; A, b, C; a, b, c; right triangles: a, b; b, c; b, B; b, A; c, A; review also § 152.

Ex. 347. State in words the first eight cases given in § 271.

PROPOSITION XLVII. PROBLEM

272. *To construct a triangle having two of its sides equal respectively to two given lines, and the angle opposite one of these lines equal to a given angle.*

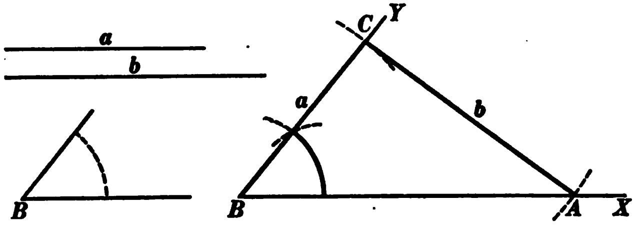


FIG. 1.

Given lines a and b, and $\angle B$.

To construct $\triangle ABC$.

I. **Construction**

1. Draw any line, as BX.

2. At any point in BX, as B, construct $\angle XBY =$ to the given $\angle B$. § 125.

3. On BY lay off $BC = a$.

4. With C as center and with b as radius, describe an arc cutting BX at A.

5. $\triangle ABC$ is the required $\triangle$.

II. The proof is left as an exercise for the student.

III. **Discussion**

(1) b may be greater than a;
(2) b may equal a;
(3) b may be less than a.

(1) If $b > a$, there will be one solution, *i.e.* one $\triangle$ and only one can be constructed which shall contain the given parts. This case is shown in Fig. 1.

(2) If $b = a$, the $\triangle$ will be isosceles. The construction will be the same as for case (1).

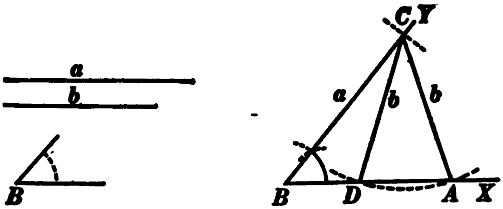


FIG. 2.

(3) If $b < a$. Make the construction as for case (1). If $b >$ the ⊥ from C to BX, there will be *two solutions*, since △ ABC and DBC, Fig. 2, both contain the required parts. If b equals the ⊥ from C to BX there will be *one* solution. The △ will be a rt. △. If $b <$ the ⊥ from C to BX, there will be *no* solution.

In the cases thus far considered, the given ∠ was acute. The discussion of the cases in which the given ∠ is a rt. ∠ and in which it is an obtuse ∠ is left to the student.

273. Question. Why is (1) the only case possible when the given angle is either right or obtuse?

274. The following exercises are given to illustrate *analysis* of problems and to show the use of auxiliary triangles in constructions.

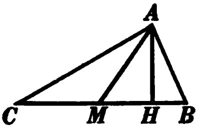


FIG. 1.

Ex. 348. Construct a triangle, given a, h_a, m_a.

ANALYSIS. Imagine the problem solved as in Fig. 1, and mark the given parts with heavy lines. The triangle AHM is determined and may be made the basis of the construction.

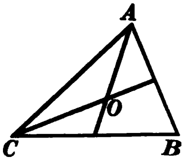


FIG. 2.

Ex. 349. Construct a triangle, given b, m_a, m_c.

ANALYSIS. From Fig. 2 it will be seen that triangle AOC may be constructed. Its three sides are known, since $AO = \frac{2}{3} m_a$ and $CO = \frac{2}{3} m_c$.

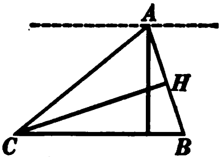


FIG. 3.

Ex. 350. Construct a triangle, given a, h_a, h_c.

ANALYSIS. In Fig. 3, right triangle CHB is determined. The locus of vertex A is a line parallel to CB, so that the distance between it and CB is equal to h_a.

Ex. 351. Construct a triangle, given b ,c, m_a.

ANALYSIS. Triangle ABK, Fig. 4, is determined by three sides, b, c, and $2m_a$. Since $ABKC$ is a parallelogram, AK bisects CB.

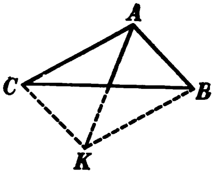


FIG. 4.

Ex. 352. Construct a triangle, given a, m_a, and the angle between m_a and a.

ANALYSIS. Triangle AMC is determined, as shown in Fig. 5.

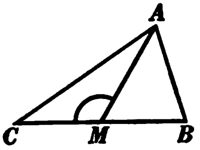


FIG. 5.

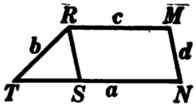


FIG. 6.

Ex. 353. Construct a trapezoid, given its four sides.

ANALYSIS. Triangle RST, Fig. 6, is determined.

Ex. 354. Construct a parallelogram, given the perimeter, one base angle, and the altitude.

CONSTRUCTION. The two parallels, CH and AE, Fig. 7, may be drawn so that the distance between them equals the altitude; at any point B construct angle EBC equal to the given base angle; draw CA, bisecting angle FCB; measure AE equal to half the given perimeter; complete parallelogram $CHEB$.

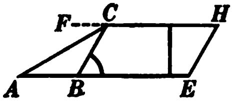


FIG. 7.

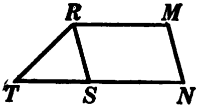


FIG. 8.

Ex. 355. Construct a trapezoid, given the two non-parallel sides and the difference between the bases. (See Fig. 8.)

Construct a triangle, given:

Ex. 356. A, h_a, t_a.

Ex. 357. h_a, l_a, d_a.

Ex. 358. h_a, m_a, l_a.

Ex. 359. a, t_b, C.

Ex. 360. B, h_a b.

Ex. 361. s_a, t_a, h_a.

Ex. 362. a, m_b, C.

Ex. 363. a, t_b, B.

Construct an isosceles trapezoid, given:

Ex. 364. The two bases and the altitude.

Ex. 365. One base, the altitude, and a diagonal.

Ex. 366. A base, a diagonal, and the angle between them.

DIRECTIONS FOR THE SOLUTION OF EXERCISES

275. I. Make the figures clear, neat, accurate, and *general* in character.

II. Fix firmly in mind hypothesis and conclusion with reference to the given figure.

III. Recall fundamental propositions related to the proposition in question.

IV. If you can find no theorem which helps you, *contradict the conclusion* in every possible way (*reductio ad absurdum*) and try to show the absurdity of the contradiction.

V. Make frequent use of the *method of analysis*, which consists in assuming the proposition proved, seeing what results follow until a known truth is reached, and then retracing the steps taken.

VI. If it is required to find a point which fulfills two conditions, it is often convenient to find the point by the *Intersection of Loci.* By finding the locus of a point which satisfies each condition separately, it is possible to find the points in which the two loci intersect; *i.e.* the points which satisfy both conditions at the same time.

VII. See § 152 and exercises following § 274 for method of attacking problems of construction.

The method just described under V is a shifting of an uncertain issue to a certain one. It is sometimes called the *Method of Successive Substitutions.* It may be illustrated thus:

1. A is true if B is true.
2. B is true if C is true.
3. But C *is* true.
4. $\therefore A$ is true.

This is also called the *Analytic Method* of proof. The proofs of the theorems are put in what is called the *Synthetic* form. But these were first thought through analytically, then rearranged in the form in which we find them.

MISCELLANEOUS EXERCISES

Ex. 367. The perpendiculars drawn from the extremities of one side of a triangle to the median upon that side are equal.

Ex. 368. Construct an angle of 75°; of $97\frac{1}{2}°$.

Ex. 369. Upon a given line find a point such that perpendiculars from it to the sides of an angle shall be equal.

Ex. 370. Construct a triangle, given its perimeter and two of its angles.

Ex. 371. Construct a parallelogram, given the base, one base angle, and the bisector of the base angle.

Ex. 372. Given two lines that would meet if sufficiently prolonged. Construct the bisector of their angle, without prolonging the lines.

Ex. 373. Construct a triangle, having given one angle, one adjacent side, and the difference of the other two sides. Case 1: The side opposite the given angle less than the other unknown side. Case 2: The side opposite the given angle greater than the other unknown side.

Ex. 374. The difference between two adjacent angles of a parallelogram is 90°; find all the angles.

Ex. 375. A straight railway passes 2 miles from a certain town. A place is described as 4 miles from the town and 1 mile from the railway. Represent the town by a point and find by construction how many places answer the description.

Ex. 376. Describe a circle through two given points which lie outside a given line, the center of the circle to be in that line. Show when no solution is possible.

Ex. 377. Construct a right triangle, given the hypotenuse and the difference of the other two sides.

Ex. 378. If two sides of a triangle are unequal, the median through their intersection makes the greater angle with the lesser side.

Ex. 379. Two trapezoids are equal if their sides taken in order are equal, each to each.

Ex. 380. Construct a right triangle, having given its perimeter and an acute angle.

Ex. 381. Draw a line such that its segment intercepted between two given indefinite lines shall be equal and parallel to a given finite line.

Ex. 382. One angle of a parallelogram is given in position and the point of intersection of the diagonals is given; construct the parallelogram.

Ex. 383. Construct a triangle, given two sides and the median to the third side.

Ex. 384. If from any point within a triangle lines are drawn to the three vertices of the triangle, the sum of these lines is less than the sum of the sides of the triangle, and greater than half their sum.

Ex. 385. Repeat the proof of Prop. XIX for two cases at once, using Figs. 1 and 2.

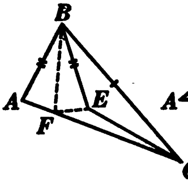


FIG. 1.

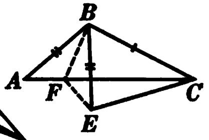


FIG. 2.

Ex. 386. If the angle at the vertex of an isosceles triangle is four times each base angle, the perpendicular to the base at one end of the base forms with one side of the triangle, and the prolongation of the other side through the vertex, an equilateral triangle.

Ex. 387. The bisector of the angle C of a triangle ABC meets AB in D, and DE is drawn parallel to AC meeting BC in E and the bisector of the exterior angle at C in F. Prove $DE = EF$.

Ex. 388. Define a locus. Find the locus of the mid-points of all the lines drawn from a given point to a given line not passing through the point.

Ex. 389. Construct an isosceles trapezoid, given the bases and one angle.

Ex. 390. Construct a square, given the sum of a diagonal and one side.

Ex. 391. The difference of the distances from any point in the base prolonged of an isosceles triangle to the equal sides of the triangle is constant.

Ex. 392. Find a point X equidistant from two intersecting lines and at a given distance from a given point.

Ex. 393. When two lines are met by a transversal, the difference of two corresponding angles is equal to the angle between the two lines.

Construct a triangle, given:

Ex. 394. A, h_a, l_a.

Ex. 395. A, t_a, s_a.

Ex. 396. a, h_a, l_a.

Ex. 397. a, $b + c$, A.

Ex. 398. a, m_a, B.

Ex. 399. m_c, h_c, B.

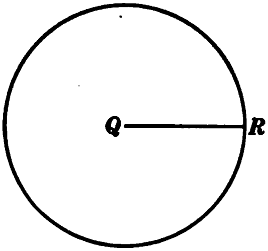
Ex. 400. b, c, $B + C$.

Ex. 401. A, B, $b + c$.

BOOK II

THE CIRCLE

276. Def. A **circle** is a plane closed figure whose boundary is a curve such that all straight lines to it from a fixed point within are equal.

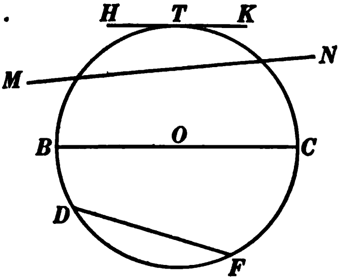


277. Def. The curve which forms the boundary of a circle is called the **circumference.**

278. Def. The fixed point within is called the **center,** and a line joining the center to any point on the circumference is called a **radius,** as *QR*.

279. From the above definitions and from the definition of equal figures, § 18, it follows that:

(*a*) *All radii of the same circle are equal.*
(*b*) *All radii of equal circles are equal.*
(*c*) *All circles having equal radii are equal.*



280. Def. Any portion of a circumference is called an **arc,** as *DF*, *FC*, etc.

281. Def. A **chord** is any straight line having its extremities on the circumference, as *DF*.

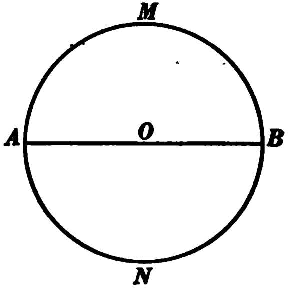
282. Def. A **diameter** is a chord which passes through the center, as *BC*.

283. Since any diameter is twice a radius, it follows that

All diameters of a circle are equal.

Proposition I. Theorem

284. ***Every diameter of a circle bisects the circumference and the circle.***



Given circle *AMBN* with center *O*, and *AB*, any diameter.

To prove: (*a*) that *AB* bisects circumference *AMBN*;
(*b*) that *AB* bisects circle *AMBN*.

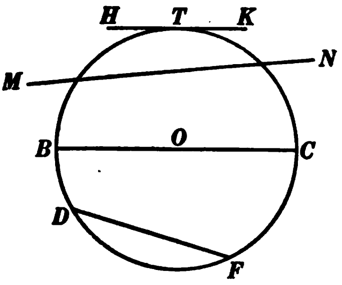
Argument	Reasons
1. Turn figure *AMB* on *AB* as an axis until it falls upon the plane of *ANB*.	1. § 54, 14.
2. Arc *AMB* will coincide with arc *ANB*.	2. § 279, *a*.
3. ∴ arc *AMB* = arc *ANB*; *i.e.* *AB* bisects circumference *AMBN*.	3. § 18.
4. Also figure *AMB* will coincide with figure *ANB*.	4. § 279, *a*.
5. ∴ figure *AMB* = figure *ANB*; *i.e.* *AB* bisects circle *AMBN*. Q.E.D.	5. § 18.

Ex. 402. A semicircle is described upon each of the diagonals of a rectangle as diameters. Prove the semicircles equal.

Ex. 403. Two diameters perpendicular to each other divide a circumference into four equal arcs. Prove by superposition.

Ex. 404. Construct a circle which shall pass through two given points.

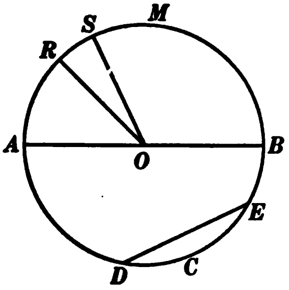
Ex. 405. Construct a circle having a given radius *r*, and passing through two given points *A* and *B*.



285. Def. A **secant** of a circle is a straight line which cuts the circumference in two points, but is not terminated by the circumference, as *MN*.

286. Def. A straight line **touches** a circle, and is a **tangent** to it if, however far prolonged, it meets the circumference in but one point. This point is called the **point of tangency**. *HK* is tangent to circle *O* at point *T*, and *T* is the point of tangency.

287. Def. A **sector** of a circle is a plane closed figure whose boundary is composed of two radii and their intercepted arc, as sector *SOR*.



288. Def. A **segment** of a circle is a plane closed figure whose boundary is composed of an arc and the chord joining its extremities, as segment *DCE*.

289. Def. A segment which is one half of a circle is called a **semicircle**, as segment *AMB*.

290. Def. An arc which is half of a circumference is called a **semicircumference**, as arc *AMB*.

291. Def. An arc greater than a semicircumference is called a **major arc**, as arc *DME*; an arc less than a semicircumference is called a **minor arc**, as arc *DCE*.

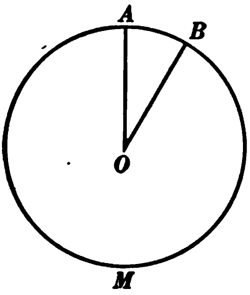
292. Def. A **central angle**, or **angle at the center**, is an angle whose vertex is at the center of a circle and whose sides are radii.

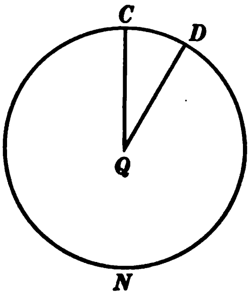
Ex. 406. The line joining the centers of two circles is 6, the radii are 8 and 10, respectively. What are the relative positions of the two circles?

Ex. 407. A circle can have only one center.

Proposition II. Theorem

293. *In equal circles, or in the same circle, if two central angles are equal, they intercept equal arcs on the circumference; conversely, if two arcs are equal, the central angles that intercept them are equal.*





I. **Given** equal circles ABM and CDN, and equal central $\angle$s O and Q, intercepting arcs AB and CD, respectively.

To prove $\overset{\frown}{AB} = \overset{\frown}{CD}$.

Argument	Reasons
1. Place circle ABM upon circle CDN so that center O shall fall upon center Q, and OA shall be collinear with QC.	1. § 54, 14.
2. A will fall upon C.	2. § 279, *b*.
3. OB will become collinear with QD.	3. By hyp.
4. $\therefore B$ will fall upon D.	4. § 279, *b*.
5. $\therefore \overset{\frown}{AB}$ will coincide with $\overset{\frown}{CD}$.	5. § 279, *b*.
6. $\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$. Q.E.D.	6. § 18.

II. Conversely:

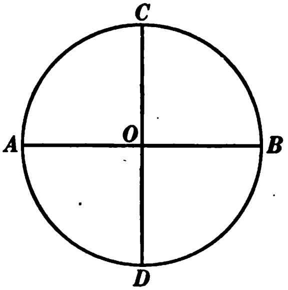
Given equal circles ABM and CDN, and equal arcs AB and CD, intercepted by $\angle$s O and Q, respectively.

To prove $\angle O = \angle Q$.

ARGUMENT	REASONS
1. Place circle *ABM* upon circle *CDN* so that center *O* shall fall upon center *Q*.	1. § 54, 14.
2. Rotate circle *ABM* upon *O* as a pivot until $\widehat{AB}$ falls upon its equal $\widehat{CD}$, *A* upon *C*, *B* upon *D*.	2. § 54, 14.
3. *OA* will coincide with *QC* and *OB* with *QD*.	3. § 17.
4. $\therefore \angle O = \angle Q$. Q.E.D.	4. § 18.

294. Cor. *In equal circles, or in the same circle, if two central angles are unequal, the greater angle intercepts the greater arc; conversely, if two arcs are unequal, the central angle that intercepts the greater arc is the greater.* (HINT. Lay off the smaller central angle upon the greater.)

295. **Def.** A fourth part of a circumference is called a **quadrant.**



From Prop. II it is evident that a right angle at the center intercepts a quadrant on the circumference. Thus, two ⊥ diameters *AB* and *CD* divide the circumference into four quadrants, *AC*, *CB*, *BD*, and *DA*.

296. **Def.** A **degree of arc,** or an **arc degree,** is the arc intercepted by a central angle of one degree.

297. A right angle contains ninety angle degrees (§ 71); therefore, since equal central angles intercept equal arcs on the circumference, a quadrant contains ninety arc degrees.

Again, four right angles contain 360 angle degrees, and four right angles at the center of a circle intercept a complete circumference; therefore, a circumference contains 360 arc degrees. Hence, a semicircumference contains 180 arc degrees.

Ex. 408. Divide a given circumference into eight equal arcs; sixteen equal arcs.

Ex. 409. Divide a given circumference into six equal arcs; three equal arcs; twelve equal arcs.

Ex. 410. A diameter and a secant perpendicular to it divide a circumference into two pairs of equal arcs.

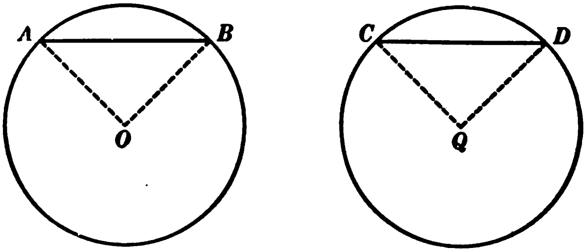
Ex. 411. Construct a circle which shall pass through two given points A and B and shall have its center in a given line c.

Ex. 412. If a diameter and another chord are drawn from a point in a circumference, the arc intercepted by the angle between them will be bisected by a diameter drawn parallel to the chord.

Ex. 413. If a diameter and another chord are drawn from a point in a circumference, the diameter which bisects their intercepted arc will be parallel to the chord.

PROPOSITION III. THEOREM

298. *In equal circles, or in the same circle, if two chords are equal, they subtend equal arcs; conversely, if two arcs are equal, the chords that subtend them are equal.*



I. **Given** equal circles O and Q, with equal chords AB and CD.

To prove $\overset{\frown}{AB} = \overset{\frown}{CD}$.

ARGUMENT	REASONS
1. Draw radii OA, OB, QC, QD.	1. § 54, 15.
2. In $\triangle$s OAB and QCD, $AB = CD$.	2. By hyp.
3. $OA = QC$ and $OB = QD$.	3. § 279, *b*.
4. $\therefore \triangle OAB = \triangle QCD$.	4. § 116.
5. $\therefore \angle O = \angle Q$.	5. § 110.
6. $\therefore \overset{\frown}{AB} = \overset{\frown}{CD}$. Q.E.D.	6. § 293, I.

II. Conversely:

Given equal circles O and Q, and equal arcs AB and CD.

To prove chord $AB =$ chord CD.

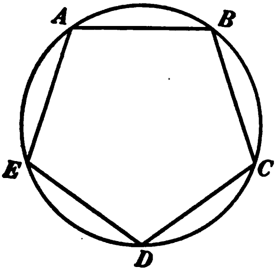
ARGUMENT	REASONS
1. Draw radii OA, OB, QC, QD.	1. § 54, 15.
2. $\overset{\frown}{AB} = \overset{\frown}{CD}$.	2. By hyp.
3. $\therefore \angle BOA = \angle DQC$.	3. § 293, II.
4. $OA = QC$ and $OB = QD$.	4. § 279, *b*.
5. $\therefore \triangle OAB = \triangle QCD$.	5. § 107
6. $\therefore$ chord $AB =$ chord CD. Q.E.D.	6. § 110.

Ex. 414. If a circumference is divided into any number of equal arcs, the chords joining the points of division will be equal.

Ex. 415. A parallelogram inscribed in a circle is a rectangle

Ex. 416. If two of the opposite sides of an inscribed quadrilateral are equal, its diagonals are equal.

Ex. 417. State and prove the converse of Ex. 416.



299. Def. A **polygon is inscribed in a circle** if all its vertices are on the circumference. Thus, polygon *ABCDE* is an inscribed polygon.

300. Def. If a polygon is inscribed in a circle, the **circle** is said to be **circumscribed about the polygon.**

Ex. 418. Inscribe an equilateral hexagon in a circle; an equilateral triangle.

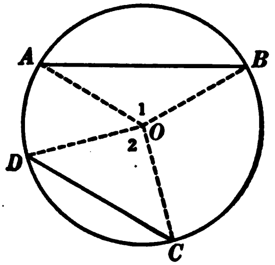
Ex. 419. The diagonals of an inscribed equilateral pentagon are equal.

Ex. 420. If the extremities of any two intersecting diameters are joined, an inscribed rectangle will be formed. Under what conditions will the rectangle be a square?

Ex. 421. State the theorems which may be used in proving arcs equal. State the theorems which may be used in proving chords equal.

PROPOSITION IV. THEOREM

301. *In equal circles, or in the same circle, if two chords are unequal, the greater chord subtends the greater minor arc; conversely, if two minor arcs are unequal, the chord that subtends the greater arc is the greater.*



I. **Given** circle O, with chord $AB >$ chord CD.

To prove $\widehat{AB} > \widehat{CD}$.

ARGUMENT	REASONS
1. Draw radii OA, OB, OC, OD.	1. § 54, 15.
2. In $\triangle$ OAB and OCD, $OA = OC$, $OB = OD$.	2. § 279, *a*.
3. Chord $AB >$ chord CD.	3. By hyp.
4. $\therefore \angle 1 > \angle 2$.	4. § 173.
5. $\therefore \widehat{AB} > \widehat{CD}$. Q.E.D.	5. § 294.

II. Conversely:

Given circle O, with $\widehat{AB} > \widehat{CD}$.

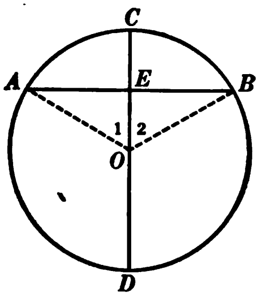
To prove chord $AB >$ chord CD.

ARGUMENT	REASONS
1. Draw radii OA, OB, OC, OD.	1. § 54, 15.
2. In $\triangle$ OAB and OCD, $OA = OC$, $OB = OD$.	2. § 279, *a*.
3. $\widehat{AB} > \widehat{CD}$.	3. By hyp.
4. $\therefore \angle 1 > \angle 2$.	4. § 294.
5. $\therefore$ chord $AB >$ chord CD. Q.E.D.	5. § 172.

Ex. 422. Prove the converse of Prop. IV by the indirect method.

PROPOSITION V. THEOREM

302. *The diameter perpendicular to a chord bisects the chord and also its subtended arcs.*



Given chord AB and diameter $CD \perp AB$ at E.

To prove $AE = EB$, $\overset{\frown}{AC} = \overset{\frown}{CB}$, and $\overset{\frown}{AD} = \overset{\frown}{DB}$.

ARGUMENT	REASONS
1. Draw radii OA and OB.	1. § 54, 15.
2. In $\triangle$ OAB, $OA = OB$.	2. § 279, *a*.
3. $\therefore$ $\triangle$ OAB is an isosceles $\triangle$.	3. § 94.
4. $\therefore$ OE bisects AB, and $AE = EB$.	4. § 212.
5. Also OE bisects $\angle$ BOA, and $\angle 1 = \angle 2$.	5. § 212.
6. $\therefore$ $\angle AOD = \angle DOB$.	6. § 75.
7. $\therefore$ $\overset{\frown}{AC} = \overset{\frown}{CB}$ and $\overset{\frown}{AD} = \overset{\frown}{DB}$. Q.E.D.	7. § 293, I.

303. Cor. I. *The perpendicular bisector of a chord passes through the center of the circle.*

304. Cor. II. *The locus of the centers of all circles which pass through two given points is the perpendicular bisector of the line which joins the points.*

305. Cor. III. *The locus of the mid-points of all chords of a circle parallel to a given line is the diameter perpendicular to the line.* (For complete proof, see p. 298.)

Ex. 423. If the diagonals of an inscribed quadrilateral are unequal, its opposite sides are unequal.

Ex. 424. Through a given point within a circle construct a chord which shall be bisected at the point.

Ex. 425. Given a line fulfilling any two of the five following conditions, prove that it fulfills the remaining three:

1. A diameter.
2. A perpendicular to a chord.
3. A bisector of a chord.
4. A bisector of the major arc of a chord.
5. A bisector of the minor arc of a chord.

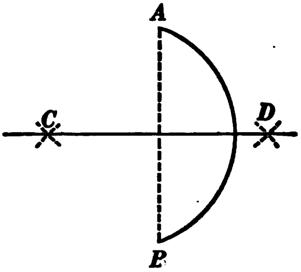
Ex. 426. Any two chords of a circle are given in position and magnitude; find the center of the circle.

Ex. 427. The line passing through the middle points of two parallel chords passes through the center of the circle.

Ex. 428. Given an arc of a circle, find the center of the circle.

PROPOSITION VI. PROBLEM

306. *To bisect a given arc.*



Given AB, an arc of any circle.

To bisect $\overset{\frown}{AB}$.

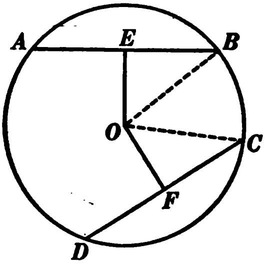
The construction, proof, and discussion are left as an exercise for the student.

Ex. 429. Construct an arc of 45°; of 30°. Construct an arc of 30°, using a radius twice as long as the one previously used. Are these two 30° arcs equal?

Ex. 430. Distinguish between finding the "mid-point of an arc" and the "center of an arc."

PROPOSITION VII. THEOREM

307. *In equal circles, or in the same circle, if two chords are equal, they are equally distant from the center; conversely, if two chords are equally distant from the center, they are equal.*



I. **Given** circle O with chord $AB =$ chord CD, and let OE and OF be the distances of AB and CD from center O, respectively

To prove $OE = OF$.

ARGUMENT	REASONS
1. Draw radii OB and OC.	1. § 54, 15.
2. E and F are the mid-points of AB and CD, respectively.	2. § 302.
3. ∴ in rt. ▵ OEB and OCF, $EB = CF$.	3. § 54, 8 *a*.
4. $OB = OC$.	4. § 279, *a*.
5. ∴ $\triangle OEB = \triangle OCF$.	5. § 211.
6. ∴ $OE = OF$. Q.E.D.	6. § 110.

II. Conversely:

Given circle O with OE, the distance of chord AB from center O, equal to OF, the distance of chord CD from center O.

To prove chord $AB =$ chord CD.

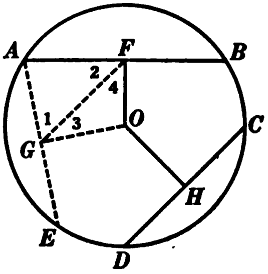
HINT. Prove $\triangle OEB = \triangle OCF$.

Ex. 431. If perpendiculars from the center of a circle to the sides of an inscribed polygon are equal, the polygon is equilateral.

Ex. 432. If through any point in a diameter two chords are drawn making equal angles with the diameter, the two chords are equal.

PROPOSITION VIII. THEOREM

308. *In equal circles, or in the same circle, if two chords are unequal, the greater chord is at the less distance from the center.*



Given circle O with chord $AB >$ chord CD, and let OF and OH be the distances of AB and CD from center O, respectively.

To prove $OF < OH$.

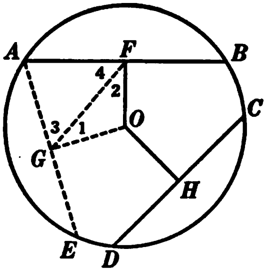
ARGUMENT	REASONS
1. From A draw a chord AE, equal to DC.	1. § 54, 15.
2. From O draw $OG \perp AE$.	2. § 155.
3. Draw FG.	3. § 54, 15.
4. $AB > CD$.	4. § By hyp.
5. $\therefore AB > AE$.	5. § 309.
6. F and G are the mid-points of AB and AE, respectively.	6. § 302.
7. $\therefore AF > AG$.	7. § 54, 8 *b*.
8. $\therefore \angle 1 > \angle 2$.	8. § 156.
9. $\angle AFO = \angle OGA$.	9. § 64.
10. $\therefore \angle 3 < \angle 4$.	10. § 54, 6.
11. $\therefore OF < OG$.	11. § 164.
12. $OG = OH$.	12. § 307, I.
13. $\therefore OF < OH$. Q.E.D.	13. § 309.

309. Note. The student should give the full statement of the substitution made; thus, reason 5 above should be: "Substituting AE for its equal CD."

PROPOSITION IX. THEOREM

(Converse of Prop. VIII)

310. *In equal circles, or in the same circle, if two chords are unequally distant from the center, the chord at the less distance is the greater.*



Given circle O with OF, the distance of chord AB from center O, less than OH, the distance of chord CD from center O.

To prove chord $AB >$ chord CD.

The proof is left as an exercise for the student.

HINT. Begin with $\triangle OGF$.

311. Cor. I. *A diameter is greater than any other chord.*

312. Cor. II. *The locus of the mid-points of all chords of a circle equal to a given chord is the circumference having the same center as the given circle, and having for radius the perpendicular from the center to the given chord.*

Ex. 433. Prove Prop. IX by the indirect method.

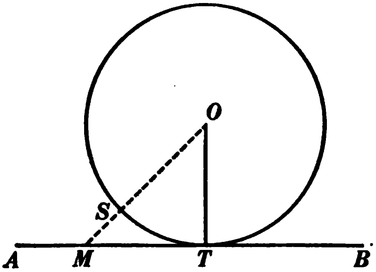
Ex. 434. Through a given point within a circle construct the minimum chord.

Ex. 435. If two chords are drawn from one extremity of a diameter, making unequal angles with it, the chords are unequal.

Ex. 436. The perpendicular from the center of a circle to a side of an inscribed equilateral triangle is less than the perpendicular from the center of the circle to a side of an inscribed square. (See § 308.)

Proposition X. Theorem

313. *A tangent to a circle is perpendicular to the radius drawn to the point of tangency.*



Given line AB, tangent to circle O at T, and OT, a radius drawn to the point of tangency.

To prove $AB \perp OT$.

Argument	Reasons
1. Let M be any point on AB other than T; then M is outside the circumference.	1. § 286.
2. Draw OM, intersecting the circumference at S.	2. § 54, 15.
3. $OS < OM$.	3. § 54, 12.
4. $OS = OT$.	4. § 279, *a*.
5. $\therefore OT < OM$.	5. § 309.
6. $\therefore OT$ is the shortest line that can be drawn from O to AB.	6. Arg. 5.
7. $\therefore OT \perp AB$; *i.e.* $AB \perp OT$. Q.E.D.	7. § 165.

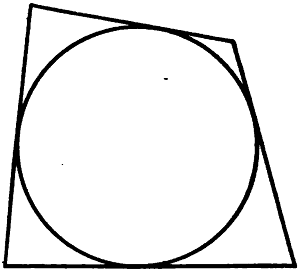
314. Cor. I. (Converse of Prop. X). *A straight line perpendicular to a radius at its outer extremity is tangent to the circle.*

Hint. Prove by the indirect method. In the figure for Prop. X, suppose that AB is not tangent to circle O at point T; then draw CD through T, tangent to circle O. Apply § 63.

315. Cor. II. *A perpendicular to a tangent at the point of tangency passes through the center of the circle.*

316. Cor. III. *A line drawn from the center of a circle perpendicular to a tangent passes through the point of tangency.*

317. Def. A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle. In the same figure the **circle** is said to be **inscribed in the polygon.**



Ex. 437. The perpendiculars to the sides of a circumscribed polygon at their points of tangency pass through a common point.

Ex. 438. The line drawn from any vertex of a circumscribed polygon to the center of the circle bisects the angle at that vertex and also the angle between radii drawn to the adjacent points of tangency.

Ex. 439. If two tangents are drawn from a point to a circle, the bisector of the angle between them passes through the center of the circle.

Ex. 440. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

Ex. 441. Tangents to a circle at the extremities of a diameter are parallel.

PROPOSITION XI. PROBLEM

318. *To construct a tangent to a circle at any given point in the circumference.*

The construction, proof, and discussion are left as an exercise for the student. (See § 314.)

Ex. 442. Construct a quadrilateral which shall be circumscribed about a circle. What kinds of quadrilaterals are circumscribable?

Ex. 443. Construct a parallelogram which shall be inscribed in a circle. What kinds of parallelograms are inscribable?

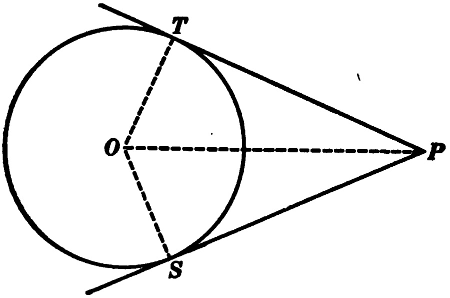
Ex. 444. Construct a line which shall be tangent to a given circle and parallel to a given line.

Ex. 445. Construct a line which shall be tangent to a given circle and perpendicular to a given line.

319. Def. The **length of a tangent** is the length of the segment included between the point of tangency and the point from which the tangent is drawn; as TP in the following figure.

PROPOSITION XII. THEOREM

320. *If two tangents are drawn from any given point to a circle, these tangents are equal.*



Given PT and PS, two tangents from point P to circle O.

To prove $PT = PS$.

The proof is left as an exercise for the student.

Ex. 446. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. 447. The median of a circumscribed trapezoid is one fourth the perimeter of the trapezoid.

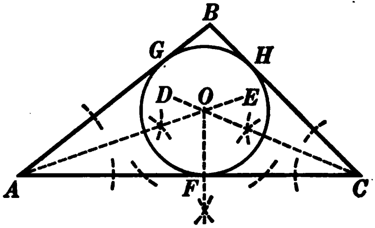
Ex. 448. A parallelogram circumscribed about a circle is either a rhombus or a square.

Ex. 449. The hypotenuse of a right triangle circumscribed about a circle is equal to the sum of the other two sides minus a diameter of the circle.

Ex. 450. If a circle is inscribed in any triangle, and if three triangles are cut from the given triangle by drawing tangents to the circle, then the sum of the perimeters of the three triangles will equal the perimeter of the given triangle.

PROPOSITION XIII. PROBLEM

321. *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

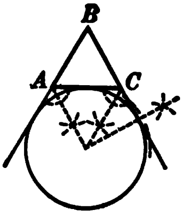
To inscribe a circle in $\triangle ABC$.

I. Construction

1. Construct AE and CD, bisecting $\angle s$ CAB and BCA, respectively. § 127.
2. AE and CD will intersect at some point as O. § 194.
3. From O draw $OF \perp AC$. § 149.
4. With O as center and OF as radius construct circle FGH.
5. Circle FGH is inscribed in $\triangle ABC$.

II. The proof and discussion are left for the student.

322. Def. A circle which is tangent to one side of a triangle and to the other two sides prolonged is said to be **escribed to the triangle.**



Ex. 451. Problem. To escribe a circle to a given triangle.

Ex. 452. (*a*) Prove that if the lines that bisect three angles of a quadrilateral meet at a common point P, then the line that bisects the remaining angle of the quadrilateral passes through P. (*b*) Tell why a circle can be inscribed in this particular quadrilateral.

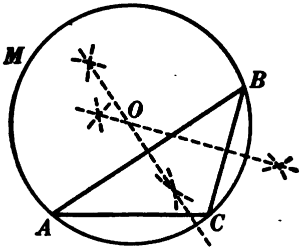
Ex. 453. In triangle ABC, draw XY parallel to BC so that

$$XY + BC = BX + CY.$$

Ex. 454. Inscribe a circle in a given rhombus.

PROPOSITION XIV. PROBLEM

323. *To circumscribe a circle about a given triangle.*



Given △ *ABC*.

To circumscribe a circle about △ *ABC*.

The construction, proof, and discussion are left as an exercise for the student.

324. **Cor.** *Three points not in the same straight line determine a circle.*

Ex. 455. Discuss the position of the center of a circle circumscribed about an acute triangle; a right triangle; an obtuse triangle.

Ex. 456. Circumscribe a circle about an isosceles trapezoid.

Ex. 457. Given the base of an isosceles triangle and the radius of the circumscribed circle, to construct the triangle.

Ex. 458. The inscribed and circumscribed circles of an equilateral triangle are concentric.

Ex. 459. If two common external tangents or two common internal tangents are drawn to two circles, the segments of these tangents intercepted between the points of contact are equal.

Ex. 460. The two segments of a secant which are between two concentric circumferences are equal.

Ex. 461. The perpendicular bisectors of the sides of an inscribed quadrilateral pass through a common point.

Ex. 462. The bisector of an arc of a circle is determined by the center of the circle and another point equidistant from the extremities of the chord of the arc.

Ex. 463. If two chords of a circle are equal, the lines which connect their mid-points with the center of the circle are equal.

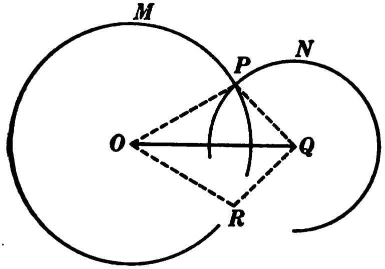
TWO CIRCLES

325. Def. The line determined by the centers of two circles is called their **line of centers** or **center-line.**

326. Def. Concentric circles are circles which have the same center.

PROPOSITION XV. THEOREM

327. *If two circumferences meet at a point which is not on their line of centers, they also meet in one other point.*



Given circumferences M and N meeting at P, a point not on their line of centers OQ.

To prove that the circumferences meet at one other point, as R.

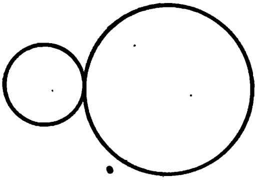
ARGUMENT	REASONS
1. Draw OP and QP.	1. § 54, 15.
2. Rotate $\triangle OPQ$ about OQ as an axis until it falls in the position QRO.	2. § 54, 14.
3. $OR = OP =$ a radius of circle M.	3. By cons.
4. $\therefore R$ is on circumference M.	4. § 279, *a*.
5. Also $QR = QP =$ a radius of circle N.	5. By cons.
6. $\therefore R$ is on circumference N.	6. § 279, *a*.
7. $\therefore R$ is on both circumference M and circumference N; *i.e.* circumferences M and N meet at R. Q.E.D.	7. Args. 4 and 6.

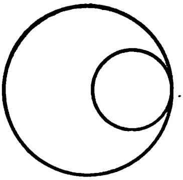
328. Cor. I. *If two circumferences intersect, their line of centers bisects their common chord at right angles.*

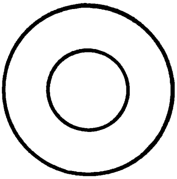
329. Cor. II. *If two circumferences meet at one point only, that point is on their line of centers.*

Hint. If they meet at a point which is not on their line of centers, they also meet in another point (§ 327). This contradicts the hypothesis.

330. Def. Two circles are said to **touch** or be **tangent to each other** if they have one and only one point in common. They are tangent **internally** or **externally** according as one circle lies within or outside of the other.


Tangent externally.


Tangent internally.


Concentric.

331. From § 330, Cor. II may be stated as follows:

If two circles are tangent to each other, their common point lies on their line of centers.

332. Cor. III. *If two circles are tangent to each other, they have a common tangent line at their point of contact.*

Hint. Apply § 314.

333. Def. A line touching two circles is called an **external common tangent** if both circles lie on the same side of it; the line is called an **internal common tangent** if the two circles lie on opposite sides of it.

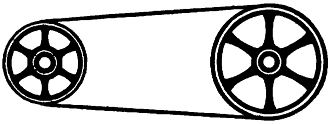


Fig. 1.

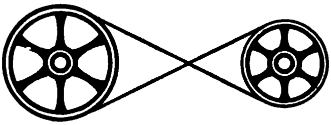


Fig. 2.

Thus a belt connecting two wheels as in Fig. 1 is an illustration of external common tangents, while a belt arranged as in Fig. 2 illustrates internal common tangents.

334. Questions. In case two circles are tangent internally how many common tangents can be drawn? in case their circumferences intersect? in case they are tangent externally? in case they are wholly outside of each other? in case one is wholly within the other?

Ex. 464. If two circles intersect, their line of centers bisects the angles between the radii drawn to the points of intersection.

Ex. 465. If the radii of two intersecting circles are 5 inches and 8 inches, what may be the length of the line joining their centers?

Ex. 466. If two circles are tangent externally, tangents drawn to them from any point in their common internal tangent are equal.

Ex. 467. Two circles are tangent to each other. Construct their common tangent at their point of contact.

Ex. 468. Construct a circle passing through a given point and tangent to a given circle at another given point.

Ex. 469. Find the locus of the centers of all circles tangent to a given circle at a given point.

MEASUREMENT

335. Def. To **measure** a quantity is to find how many times it contains another quantity of the same kind. The result of the measurement is a **number** and is called the **numerical measure**, or **measure-number**, of the quantity which is measured. The measure employed is called the **unit of measure.**

Thus, the length or breadth of a room is measured by finding how many feet there are in it; *i.e.* how many times it contains a foot as a measure.

336. It can be shown that to *every* geometric magnitude there corresponds a definite number called its measure-number. The proof that *to every straight line segment there belongs a measure-number* is found in the Appendix, § 595. The method of proof there used shows that operations with measure-numbers follow the ordinary laws of algebra.

337. Def. Two quantities are **commensurable** if there exists a measure that is contained an integral number of times in each. Such a measure is called a **common measure** of the two quantities.

Thus, a yard and a foot are commensurable, each containing an inch a whole number of times; so, too, $12\frac{1}{2}$ inches and $18\frac{3}{4}$ inches are commensurable, each containing a fourth of an inch a whole number of times.

338. Questions. If two quantities have a common measure, *how many* common measures have they? Name some common measures of $12\frac{1}{2}$ inches and $18\frac{3}{4}$ inches. What is their greatest common measure? What is their least common measure?

339. Def. Two quantities are **incommensurable** if there exists no measure that is contained an integral number of times in each.

It will be shown later that a diagonal and a side of the same square cannot be measured by the same unit, without a remainder; and that the diagonal is equal to $\sqrt{2}$ times the numerical measure of the side. Now $\sqrt{2}$ can be expressed only approximately as a simple fraction or as a decimal. It lies between 1.4 and 1.5, for $(1.4)^2 = 1.96$, and $(1.5)^2 = 2.25$. Again, it lies between 1.41 and 1.42,* between 1.414 and 1.415, between 1.4142 and 1.4143, and so on. By repeated trials values may be found approximating more and more closely to $\sqrt{2}$, but no decimal number can be obtained that, taken twice as a factor, will give exactly 2.

340. When we speak of the *ratio* of one quantity to another, we have in mind their *relative* sizes. By this is meant not the *difference* between the two, but *how many times* one contains the other or some aliquot part of it. In algebra the *ratio of two numbers* has been defined as the indicated quotient of the first divided by the second. Since to each geometric magnitude there corresponds a number called its measure-number (§ 336), therefore:

341. Def. The **ratio of two geometric magnitudes** may be defined as the quotient of their measure-numbers, when the same measure is applied to each.

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* The student should multiply to get the successive approximations.

Thus, if the length of a room is 36 feet and the width 27 feet, the ratio of the length to the width is said to be the ratio of 36 to 27; *i.e.* $\frac{36}{27}$, which is equal to $\frac{4}{3}$. The ratio of the width to the length is $\frac{27}{36}$, which is equal to $\frac{3}{4}$. The term ratio is never applied to two magnitudes that are unlike.

342. Def. If the two magnitudes compared are commensurable, the ratio is called a **commensurable ratio** and can always be expressed as a simple fraction.

343. Def. If the two magnitudes compared are incommensurable, the ratio is called an **incommensurable ratio** and can be expressed only approximately as a simple fraction. Closer and closer approximations to an incommensurable ratio may be obtained by repeatedly using smaller and smaller units as measures of the two magnitudes to be compared and by finding the quotient of the numbers thus obtained.

Two magnitudes, *e.g.* two line segments, taken at random are usually incommensurable, commensurability being comparatively rare.

344. Historical Note. The discovery of incommensurable magnitudes is ascribed to Pythagoras, whose followers for a long time kept the discovery a secret. It is believed that Pythagoras was the first to prove that the side and diagonal of a square are incommensurable. A more complete account of the work of Pythagoras will be found in § 510.

Ex. 470. What is the greatest common measure of 48 inches and 18 inches? Will it divide 48 inches − 18 inches? 48 inches − 2 × 18 inches?

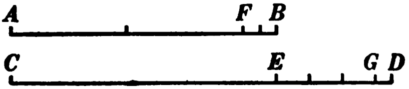
Ex. 471. Draw any two line segments which have a common measure. Find the sum of these lines and, by laying off the common measure, show that it is a measure of the sum of the lines.

Ex. 472. Given two lines, 5 inches and 4 inches long, respectively. Show by a diagram that any common measure of 5 inches and 4 inches is also a measure of 15 inches plus or minus 8 inches.

Ex. 473. Find the greatest common divisor of 728 and 844 by division and point out the similarity of the process to that used in Prop XVI

PROPOSITION XVI. PROBLEM

345. *To determine whether two given lines are commensurable or not; and if they are commensurable, to find their common measure and their ratio.*



Given lines AB and CD.

To determine: (*a*) whether AB and CD are commensurable and if so,
(*b*) what is their common measure; and
(*c*) what is the ratio of AB to CD.

I. Construction

1. Measure off AB on CD as many times as possible. Suppose it is contained once, with a remainder ED.

2. Measure off ED on AB as many times as possible. Suppose it is contained twice, with a remainder FB.

3. Measure off FB on ED as many times as possible. Suppose it is contained three times, with a remainder GD.

4. Measure off GD on FB as many times as possible, and so on.

5. It is evident that this process will terminate only when a remainder is obtained which is a measure of the remainder immediately preceding.

6. If this process terminates, then the two given lines are commensurable, and the last remainder is their greatest common measure.

7. For example, if GD is a measure of FB, then AB and CD are commensurable, GD is their greatest common measure, and the ratio of AB to CD can be found.

II. Proof

ARGUMENT	REASONS
1. Suppose $FB = 2\ GD$.	1. See I, 7.
2. $ED = EG + GD$.	2. § 54, 11.

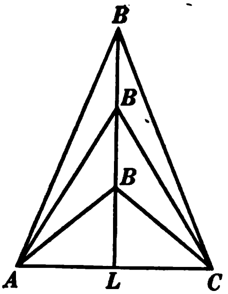
3. ∴ $ED = 3\,FB + GD = 7\,GD$.	3. § 309.
4. $AB = 2\,ED + FB$.	4. § 54, 11.
5. ∴ $AB = 14\,GD + 2\,GD = 16\,GD$.	5. § 309.
6. $CD = AB + ED$.	6. § 54, 11.
7. ∴ $CD = 16\,GD + 7\,GD = 23\,GD$.	7. § 309.
8. ∴ AB and CD are commensurable.	8. § 337.
9. Also, GD is a common measure of AB and CD.	9. Args. 5 and 7.
10. The ratio of AB to CD = the ratio of $16\,GD$ to $23\,GD = \frac{16}{23}$. Q.E.D.	10. § 341.

III. A full discussion of this problem will be found in the Appendix, § 598.

CONSTANTS AND VARIABLES. LIMITS

346. Consider an isosceles triangle ABC, whose base is AC and whose altitude is LB. Keeping the base AC the same (*constant*), suppose the altitude to change (*vary*).

If LB increases, what will be the effect upon the lengths of AB and CB? what the effect upon the base angles? upon the vertex angle? Will the base angles always be *equal* to each other? What *limiting* value have they? Is the base angle *related* to half the vertex angle or are the two *independent*? What relation is there? Is this relation constant or does it change?



Imagine the altitude of the triangle to diminish. Repeat the questions given above, considering the altitude as decreasing. What is now the limiting value for the altitude? what for the length of one of the equal sides? for the base angles? for the angle at the vertex?

Ex. 474. Consider an isosceles triangle with a constant altitude and a variable base. Repeat the questions given above.

Ex. 475. Consider an isosceles triangle with constant base angles, but variable base. Tell what other constants and what other variables there would be in this case.

Ex. 476. If through any point in the base of an isosceles triangle lines are drawn parallel to the equal sides of the triangle, a parallelogram will be formed whose perimeter will be *constant*; *i.e.* the perimeter will be independent of the position of the point.

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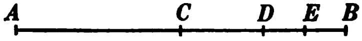
347. Def. A magnitude is **constant** if it does not change throughout a discussion.

348. Def. A magnitude is **variable** if it takes a series of different successive values during a discussion.

349. Def. If a variable approaches a constant in such a way that the difference between the variable and the constant may be made to become and remain smaller than any fixed number previously assigned, however small, the constant is called the **limit** of the variable.

350. The variable is said to **approach** its limit as it becomes more and more nearly equal to it. Thus, suppose a point to move from A toward B, by successive steps, under the restriction that at each step it must go over one half the segment between it and B. At the first step it reaches C, whereupon there remains the segment CB to be traveled over; at the next step it reaches D, and there remains an equal segment to be covered. Whatever the number of steps taken, there must always *remain* a segment equal to the segment last covered. But the segment between A and the moving point may be made to differ from AB by *as little as we please*, *i.e.* by less than *any previously assigned value.* For assign some value, say, half an inch. Then the point, continuing to move under its governing law, may approach B until there remains a segment less than half an inch. Whatever be the value assigned, the variable segment from A to the moving point may be made to differ from the constant segment AB by less than the assigned value.



Again, the numbers in the series 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., in which each term is one-half of the preceding term, approach 0 as a limit as the number of terms in the series is increased. For if we assign any value, as $\frac{1}{10000}$, it is evident that a term of the series may be found which is less than $\frac{1}{10000}$; it is also evident that no term of the series can become 0.

351. In elementary geometry the variables that approach limits are usually such that they cannot attain their limits. There are, however, variables that do attain their limits. The limiting values of algebraic expressions are frequently of this kind; thus, the expression $\frac{1}{x^2+1}$ approaches 1 as x approaches 0, and has the limit 1 when x becomes zero.

352. Def. Two variables are said to be **related** when one depends upon the other so that, if the value of one variable is known, the value of the other can be obtained.

For example, the diagonal and the area of a square are related variables, for there is a value for the area for any value which may be given to the diagonal, and *vice versa.*

353. Questions. On the floor is a bushel of sand. If we keep adding to this pile forever, how large will it become? Does it depend upon the law governing our additions? If we add one quart each hour, how large will it become? If we add one quart the first hour, a half quart the second hour, a fourth quart the third hour, etc., each hour adding one half as much as the preceding hour, how large will the pile become?

354. Historical Note. Achilles and the Tortoise. One of the early Greek schools of mathematics, founded during the fifth century B.C., at Elea, Italy, and known as the Eleatic School, was famous for its investigations of problems involving infinite series. Zeno, one of the most prominent members, proposed this question: He "argued that if Achilles ran ten times as fast as a tortoise, yet if the tortoise had (say) 1000 yards start, it could never be overtaken: for, when Achilles had gone the 1000 yards, the tortoise would still be 100 yards in front of him; by the time he had covered these 100 yards, it would still be 10 yards in front of him; and so on forever: thus Achilles would get nearer and nearer to the tortoise but never overtake it." Was Zeno right? If not, can you find the fallacy in his argument?

PROPOSITION XVII. THEOREM

355. *If two variables are always equal, and if each approaches a limit, then their limits are equal.*

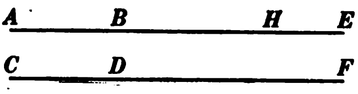
Given two variables, V and V', which are always equal and which approach as limits L and L', respectively.

To prove $L = L'$.

ARGUMENT	REASONS
1. Either $L = L'$, or $L \neq L'$.	1. § 161, *a*.
2. Suppose that one limit is greater than the other, say $L > L'$; then V, in approaching L, may assume a value between L' and L; *i.e.* V may assume a value $> L'$.	2. § 349.
3. But V' cannot assume a value $> L'$.	3. § 349.
4. $\therefore$ V may become $> V'$.	4. Args. 2 and 3.
5 But this is impossible, since V and V' are always equal.	5. By hyp.
6. $\therefore$ $L = L'$. Q.E.D.	6. § 161, *b*.

356. Question. In the above proof are V and V' increasing or decreasing variables? The student may adapt the argument above to the case in which V and V' are decreasing variables.

Ex. 477. Apply Prop. XVII to the accompanying figure, where variable V is represented by the line AB, variable V' by the line CD, limit L by line AE, and limit L' by line CF.

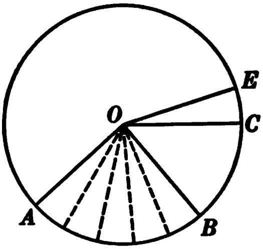


357. Note. It will be seen that, in the application of Prop. XVII, there are three distinct things to be considered:

(1 Two variables that are *always equal;*
(2) The *limits* of these two variables;
(3) The *equality* of these limits themselves.

Proposition XVIII. Theorem

358. *An angle at the center of a circle is measured by its intercepted arc.*



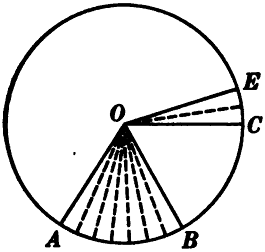
Given central $\angle AOB$ and $\overset{\frown}{AB}$ intercepted by it; let $\angle COE$ be any unit $\angle$ (*e.g.* a degree), and let $\overset{\frown}{CE}$, intercepted by the unit $\angle$, be the unit arc.

To prove the measure-number of $\angle AOB$, referred to $\angle COE$, equal to the measure-number of $\overset{\frown}{AB}$, referred to $\overset{\frown}{CE}$.

I. If $\angle AOB$ and $\angle COE$ are commensurable.

(*a*) Suppose that $\angle COE$ is contained in $\angle AOB$ an integral number of times.

Argument	Reasons
1. Apply $\angle COE$ to $\angle AOB$ as a measure. Suppose that $\angle COE$ is contained in $\angle AOB$ r times.	1. § 335.
2. Then r is the measure-number of $\angle AOB$ referred to $\angle COE$ as a unit.	2. § 335.
3 Now the r equal central $\angle$s which compose $\angle AOB$ intercept r equal arcs on the circumference, each equal to $\overset{\frown}{CE}$.	3. § 293, I.
4. $\therefore$ r is the measure-number of $\overset{\frown}{AB}$ referred to $\overset{\frown}{CE}$ as a unit.	4. § 335.
5. $\therefore$ the measure-number of $\angle AOB$, referred to $\angle COE$ as a unit, equals the measure-number of $\overset{\frown}{AB}$, referred to $\overset{\frown}{CE}$ as a unit. Q.E.D.	5. § 54, 1.

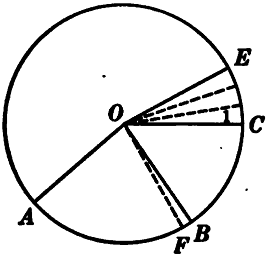


I. If ∠ *AOB* and ∠ *COE* are commensurable.

(*b*) Suppose that ∠ *COE* is not contained in ∠ *AOB* an integral number of times. The proof is left as an exercise for the student.

Hint. Some aliquot part of ∠*COE* must be a measure of ∠*AOB* (Why?) Try ½ ∠*COE*, ⅓ ∠*COE*, etc.

II. If ∠ *AOB* and ∠ *COE* are incommensurable.



Argument	Reasons
1. Let ∠ 1 be a measure of ∠ *COE*. Apply ∠ 1 to ∠ *AOB* as many times as possible. There will then be a remainder, ∠*FOB*, less than ∠1.	1. § 339.
2. ∠ *AOF* and ∠ *COE* are commensurable.	2. § 337.
3. ∴ the measure-number of ∠ *AOF*, referred to ∠ *COE* as a unit, equals the measure-number of $\widehat{AF}$, referred to $\widehat{CE}$ as a unit.	3. § 358, I.

Argument	Reasons
4. Now take a smaller measure of $\angle COE$. No matter how small a measure of $\angle COE$ is taken, when it is applied as a measure to $\angle AOB$, the remainder, $\angle FOB$, will be smaller than the ∠ taken as a measure.	4. § 335.
5. Also $\widehat{FB}$ will be smaller than the arc intercepted by the ∠ taken as a measure.	5. § 294.
6. ∴ the difference between $\angle AOF$ and $\angle AOB$ may be made to become and remain less than any previously assigned ∠, however small; and likewise the difference between $\widehat{AF}$ and $\widehat{AB}$, less than the arc intercepted by the assigned ∠.	6. Args. 4 and 5.
7. ∴ $\angle AOF$ approaches $\angle AOB$ as a limit, and $\widehat{AF}$ approaches $\widehat{AB}$ as a limit.	7. § 349.
8. Hence the measure-number of $\angle AOF$ approaches the measure-number of $\angle AOB$ as a limit, and the measure-number of $\widehat{AF}$ approaches the measure-number of $\widehat{AB}$ as a limit.	8. § 359.
9. But the measure-number of $\angle AOF$ is always equal to the measure-number of $\widehat{AF}$.	9. Arg. 3.
10. ∴ the measure-number of $\angle AOB$, referred to $\angle COE$ as a unit, equals the measure-number of $\widehat{AB}$, referred to $\widehat{CE}$ as a unit. Q.E.D.	10. § 355.

359. *If a magnitude is variable and approaches a limit, then, as the magnitude varies, the successive measure-numbers of the variable approach as their limit the measure-number of the limit of the magnitude.*

(This theorem will be found in the Appendix, § 597.)

360. Cor. *In equal circles, or in the same circle, two angles at the center have the same ratio as their intercepted arcs.*

HINT. The *measure-numbers* of the angles are equal respectively to the measure-numbers of their intercepted arcs. Therefore the ratio of the angles is equal to the ratio of the arcs.

Ex. 478. Construct a secant which shall cut off two thirds of a given circumference.

Ex. 479. Is the ratio of two chords in the same circle equal to the ratio of the arcs which they subtend? Illustrate your answer, using a semicircumference and a quadrant.

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361. The symbol $\underline{\propto}$ will be used for *is measured by.* $\propto$ is the symbol of variation, and the macron (–) means long or length. Hence $\underline{\propto}$ suggests *varies as the length of.*

362. From § 336 it follows directly that:

(*a*) *In equal circles, or in the same circle, equal angles are measured* by equal arcs; conversely, equal arcs measure equal angles.*

(*b*) *The measure of the* $\left\{\begin{matrix}\textit{sum}\\ \textit{difference}\end{matrix}\right\}$ *of two angles* **is** *equal to the* $\left\{\begin{matrix}\textit{sum}\\ \textit{difference}\end{matrix}\right\}$ *of the measures of the angles.*

(*c*) *The measure of any multiple of an angle* **is equal to that** *same multiple of the measure of the angle.*

363. Def An **angle** is said to be **inscribed in a circle** if its vertex lies on the circumference and its sides are chords.

364. Def. An **angle** is said to be **inscribed in a segment of a circle** if its vertex lies on the arc of the segment and its sides pass through the extremities of that arc.

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* It is, of course, inaccurate to speak of measuring one magnitude by another magnitude of a different kind; but, in this case, it has become a convention so general that the student needs to become familiar with it. More accurately, in Prop. XVIII, the *measure-number* of an angle at the center, referred to any unit angle, is the same as the *measure number* of its intercepted arc when the unit arc is the arc intercepted by the unit angle.

PROPOSITION XIX. THEOREM

365. *An inscribed angle is measured by one half its intercepted arc.*

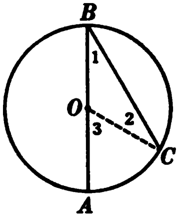


FIG. 1.

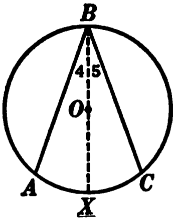


FIG. 2.

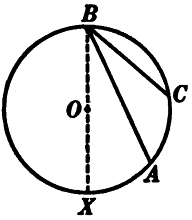


FIG. 3.

Given inscribed $\angle ABC$.

To prove that $\angle ABC \mathrel{\underline{\propto}} \frac{1}{2}\overset{\frown}{AC}$.

I. Let one side of $\angle ABC$, as AB, pass through the center of the circle (Fig. 1).

ARGUMENT	REASONS
1. Draw radius OC.	1. § 54, 15.
2. $OB = OC$.	2. § 279, *a*.
3. $\therefore \angle 1 = \angle 2$.	3. § 111.
4. $\angle 1 + \angle 2 = \angle 3$.	4. § 215.
5. $\therefore 2\angle 1 = \angle 3$.	5. § 309.
6. $\therefore \angle 1 = \frac{1}{2}\angle 3$.	6. § 54, 8 *a*.
7. But $\angle 3 \mathrel{\underline{\propto}} \overset{\frown}{AC}$.	7. § 358.
8. $\therefore \frac{1}{2}\angle 3$ or $\angle 1 \mathrel{\underline{\propto}} \frac{1}{2}\overset{\frown}{AC}$; *i. e.* $\angle ABC \mathrel{\underline{\propto}} \frac{1}{2}\overset{\frown}{AC}$. Q.E.D.	8. § 362, *c*.

II. Let center O lie within $\angle ABC$ (Fig. 2).

III. Let center O lie outside of $\angle ABC$ (Fig. 3).

The proofs of II and III are left as exercises for the student.

HINT. In Fig. 2, what is the measure of $\angle 4$? of $\angle 5$? What, then, is the measure of $\angle ABC$? In Fig. 3, what is the measure of $\angle XBC$? of $\angle XBA$? What, then, is the measure of $\angle ABC$?

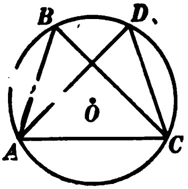


FIG. 1.

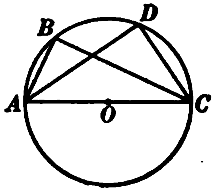


FIG. 2.

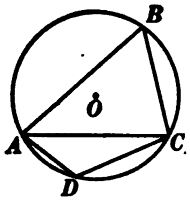


FIG. 3.

366. **Cor. I.** *All angles inscribed in the same segment are equal.* (See Fig. 1.)

367. **Cor. II.** *Any angle inscribed in a semicircle is a right angle.* (See Fig. 2.)

368. **Cor. III.** *The locus of the vertex of a right triangle having a given hypotenuse as base is the circumference having the same hypotenuse as diameter.*

369. **Cor. IV.** *Any angle inscribed in a segment less than a semicircle is an obtuse angle.* (See Fig. 3.)

370. **Cor. V.** *Any angle inscribed in a segment greater than a semicircle is an acute angle.* (See Fig. 3.)

Ex. 480. If an inscribed angle contains 24 angle degrees, how many arc degrees are there in the intercepted arc? how many in the rest of the circumference?

Ex. 481. If an inscribed angle intercepts an arc of 70°, how many degrees are there in the angle?

Ex. 482. How many degrees are there in an angle inscribed in a segment whose arc is 140°?

Ex. 483. Construct any segment of a circle so that an angle inscribed in it shall be an angle of: (*a*) 60°; (*b*) 45°; (*c*) 30°.

Ex. 484. Repeat Ex. 483, using a given line as chord of the segment. How many solutions are there to each case of Ex. 483? how many to each case of Ex. 484?

Ex. 485. The opposite angles of an inscribed quadrilateral are supplementary.

Ex. 486. **If the diameter of a circle is one of the equal sides of an isosceles triangle, the circumference will bisect the base of the triangle.**

Ex. 487. By means of a circle construct a right triangle, given the hypotenuse and an arm.

Ex. 488. By means of a circle construct a right triangle, given the hypotenuse and an adjacent angle

Ex. 489. Construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.

---

371. Historical Note. Thales (640–546 B.C.), the founder of the earliest Greek school of mathematics, is said to have discovered that all triangles having a diameter of a circle as base, with their vertices on the circumference, have their vertex angles right angles. Thales was one of the Seven Wise Men. He had much business shrewdness and sagacity, and was renowned for his practical and political ability. He went to Egypt in his youth, and while there studied geometry and astronomy. The story is told that one day while viewing the stars, he fell into a ditch; whereupon an old woman said, "How canst thou know what is doing in the heavens, when thou seest not what is at thy feet?"

THALES

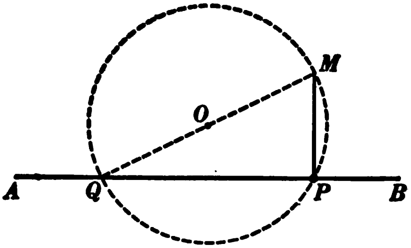
According to Plutarch, Thales computed the height of the Pyramids of Egypt from measurements of their shadows. Plutarch gives a dialogue in which Thales is addressed thus, "Placing your stick at the end of the shadow of the pyramid, you made by the same rays two triangles, and so proved that the height of the pyramid was to the length of the stick as the shadow of the pyramid to the shadow of the stick." This computation was regarded by the Egyptians as quite remarkable, since they were not familiar with applications of abstract science.

The geometry of the Greeks was in general ideal and speculative, the Greek mind being more attracted by beauty and by abstract relations than by the practical affairs of everyday life.

PROPOSITION XX. PROBLEM

372. *To construct a perpendicular to a given straight line at a given point in the line.*

(Second method. For another method, see § 148.)



Given line AB and P, a point in AB.

To construct a ⊥ to AB at P.

I. **Construction**

1. With O, any convenient point outside of AB, as center, and with OP as radius, construct a circumference cutting AB at P and Q.
2. Draw diameter QM.
3. Draw PM.
4. PM is ⊥ AB at P.

II. The proof and discussion are left as an exercise for the student.

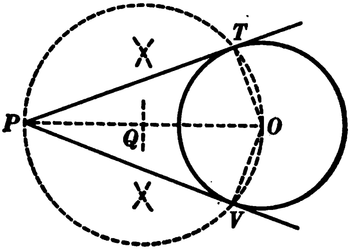
The method of § 372 is useful when the point P is at or near the end of the line AB.

Ex. 490. Construct a perpendicular to line AB at its extremity B, without prolonging AB.

Ex. 491. Through one of the points of intersection of two circumferences a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.

PROPOSITION XXI. PROBLEM

373. *To construct a tangent to a circle from a point outside.*



Given circle O and point P outside the circle.

To construct a tangent from P to circle O.

I. Construction

1. Draw PO.

2. With Q, the mid-point of PO, as center, and with QO as radius, construct a circumference intersecting the circumference of circle O in points T and V.

3. Draw PT and PV.

4. PT and PV are tangents from P to circle O.

II. The proof and discussion are left as an exercise for the student.

HINT. Draw OT and OV and apply § 367.

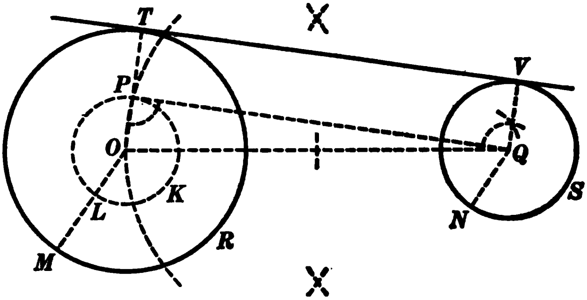
Ex. 492. Circumscribe an isosceles triangle about a given circle, the base of the isosceles triangle being equal to a given line. What restriction is there on the length of the base?

Ex. 493. Circumscribe a right triangle about a given circle, one arm of the triangle being equal to a given line. What is the least length possible for the given line, as compared with the diameter of the circle?

Ex. 494. If a circumference M passes through the center of a circle B, the tangents to B at the points of intersection of the circles intersect on circumference M.

Ex. 495. Circumscribe an isosceles triangle about a circle, the altitude upon the base of the triangle being given.

Ex. 496. Construct a common external tangent to two given circles.



Given circles MTR and NVS.

To construct a common external tangent to circles MTR and NVS.

I. Construction

1. Draw the line of centers OQ.

2. Suppose radius $OM >$ radius QN. Then, with O as center and with $OL = OM - QN$ as radius, construct circle LPK.

3. Construct tangent QP from point Q to circle LPK. § 373.

4. Draw OP and prolong it to meet circumference MTR at T.

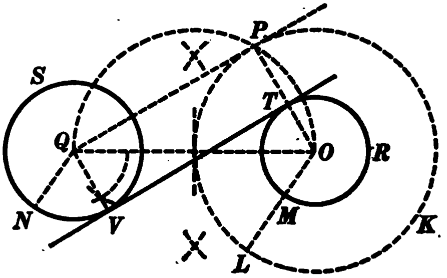
5. Draw $QV \parallel OT$. § 188.

6. Draw TV.

7. TV is tangent to circles MTR and NVS.

II. The proof and discussion are left as an exercise for the student.

Ex. 497. Construct a second common external tangent to circles MTR and NVS by same method. How is the method modified if the two circles are equal?



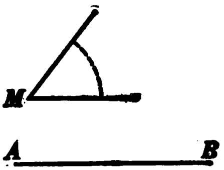
Ex. 498. Construct a common internal tangent to two circles.

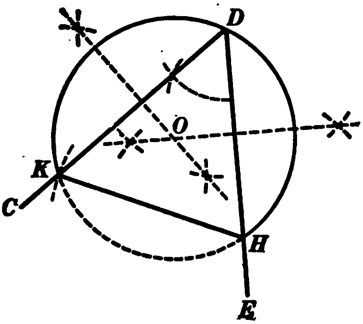
HINT. Follow steps of Ex. 496 except step 2. Make $OL = OM + QN$.

Ex. 499. By moving Q toward O in the preceding figure, show when there are four common tangents; when only three; when only two; when only one; when none.

PROPOSITION XXII. PROBLEM

374. *With a given line as chord, to construct a segment of a circle capable of containing a given angle.*





Given line AB and $\angle M$.

To construct, with AB as chord, a segment of a circle capable of containing $\angle M$.

I. Construction

1. Construct an $\angle$, as $\angle CDE$, equal to the given $\angle M$. § 125.
2. With H, any convenient point on DE, as center and with AB as radius, describe an arc cutting DC at some point, as K.
3. Draw HK.
4. Circumscribe a circle about $\triangle KDH$. § 323.
5. Segment KDH is the segment capable of containing $\angle M$.

II. The proof and discussion are left to the student.

375. Questions. Without moving AB, can you construct a $\triangle$ with AB as base and a vertex $\angle = \angle M$? What is the sum of the base $\angle$s?

376. Cor. *The locus of the vertices of all triangles having a given base and a given angle at their vertices is the arc which forms, with the given base, a segment capable of containing the given angle.*

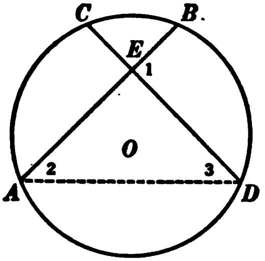
Ex. 500. On a given line construct a segment that shall contain an angle of 105°; of 135°.

Ex. 501. Find the locus of the vertices of all triangles having a common base 2 inches long and having their vertex angles equal to 60°.

Ex. 502. Construct a triangle, having given b, h_b, and B.

PROPOSITION XXIII. THEOREM

377. ***An angle formed by two chords which intersect within a circle is measured by one half the sum of the arc intercepted between its sides and the arc intercepted between the sides of its vertical angle.***



Given two chords AB and CD, intersecting at E.

To prove that $\angle 1 \underline{\propto} \frac{1}{2}(\overset{\frown}{BD} + \overset{\frown}{AC})$.

ARGUMENT	**REASONS**
1. Draw AD.	1. § 54, 15.
2. $\angle 1 = \angle 2 + \angle 3$.	2. § 215.
3. $\angle 2 \underline{\propto} \frac{1}{2} \overset{\frown}{BD}$.	3. § 365.
4. $\angle 3 \underline{\propto} \frac{1}{2} \overset{\frown}{AC}$.	4. § 365.
5. $\therefore \angle 2 + \angle 3 \underline{\propto} \frac{1}{2}(\overset{\frown}{BD} + \overset{\frown}{AC})$.	5. § 362, b.
6. $\therefore \angle 1 \underline{\propto} \frac{1}{2}(\overset{\frown}{BD} + \overset{\frown}{AC})$. Q.E.D.	6. § 309.

Ex. 503. One angle formed by two intersecting chords intercepts an arc of 40°. Its vertical angle intercepts an arc of 60°. How large is the angle?

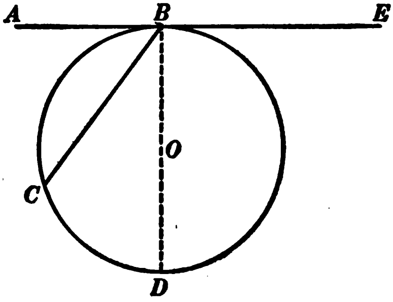
Ex. 504. If an angle of two intersecting chords is 40° and its intercepted arc is 30°, how large is the opposite arc?

Ex. 505. If two chords intersect at right angles within a circumference, the sum of two opposite intercepted arcs is equal to a semicircumference.

Ex. 506. If M is the center of a circle inscribed in triangle ABC and if AM is prolonged to meet the circumference of the circumscribed circle at D, prove that $BD = DM = DC$.

PROPOSITION XXIV. THEOREM

378. *An angle formed by a tangent and a chord is measured by one half its intercepted arc.*



Given $\angle ABC$ formed by tangent AB and chord BC.

To prove that $\angle ABC \overset{\propto}{=} \frac{1}{2}\,\overset{\frown}{BC}$.

ARGUMENT	REASONS
1. Draw diameter BD.	1. § 54, 15.
2. $\angle ABD$ is a rt. $\angle$.	2. § 313.
3. $\therefore \angle ABD \overset{\propto}{=} \frac{1}{2}$ a semicircumference; *i.e.* $\angle ABD \overset{\propto}{=} \frac{1}{2}$ arc BCD.	3. § 297.
4. $\angle CBD \overset{\propto}{=} \frac{1}{2}\,\overset{\frown}{CD}$.	4. § 365.
5. $\therefore \angle ABD - \angle CBD \overset{\propto}{=} \frac{1}{2}$ (arc $BCD - \overset{\frown}{CD}$).	5. § 362, *b*.
6. $\therefore \angle ABC \overset{\propto}{=} \frac{1}{2}\,\overset{\frown}{BC}$. Q.E.D.	6. § 309.

Ex. 507. In the figure of § 378, if arc $BC = 100°$, find the number of degrees in angle ABC; in angle CBD; in angle CBE.

Ex. 508. If tangents are drawn at the extremities of a chord which subtends an arc of 120°, what kind of triangle is formed?

Ex. 509. If a tangent is drawn to a circle at the extremity of a chord, the mid-point of the subtended arc is equidistant from the chord and the tangent.

Ex. 510. Solve Prop. XXII by means of Prop. XXIV.

HINT. Observe that in the figure for Prop. XXIV any angle inscribed in segment BDC would be equal to angle ABC.

PROPOSITION XXV. THEOREM

379. *An angle formed by two secants intersecting outside of a circumference, an angle formed by a secant and a tangent, and an angle formed by two tangents are each measured by one half the difference of the intercepted arcs.*

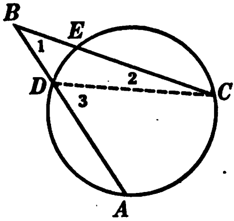


FIG. 1.

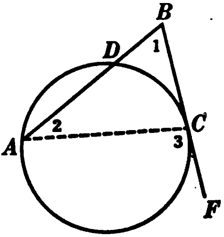


FIG. 2.

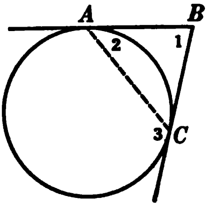


FIG. 3.

I. An angle formed by two secants (Fig. 1).

Given two secants BA and BC, forming $\angle 1$.

To prove that $\angle 1 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$.

ARGUMENT	REASONS
1. Draw CD.	1. § 54, 15.
2. $\angle 1 + \angle 2 = \angle 3$.	2. § 215.
3. $\therefore \angle 1 = \angle 3 - \angle 2$.	3. § 54, 3.
4. $\angle 3 \overset{\propto}{=} \frac{1}{2}\overset{\frown}{AC}$, and $\angle 2 \overset{\propto}{=} \frac{1}{2}\overset{\frown}{DE}$.	4. § 365.
5. $\therefore \angle 3 - \angle 2 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$.	5. § 362, *b*.
6. $\therefore \angle 1 \overset{\propto}{=} \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{DE})$. Q.E.D.	6. § 309.

II. An angle formed by a secant and a tangent (Fig. 2).

III. An angle formed by two tangents (Fig. 3).

The proofs of II and III are left to the student.

380. Note. In the preceding theorems the vertex of the angle may be: (1) within the circle; (2) on the circumference; (3) outside the circle.

Ex. 511. Tell how to measure an angle having its vertex in each of the three possible positions with regard to the circumference.

PROPOSITION XXVI. THEOREM

381. *Parallel lines intercept equal arcs on a circumference.*

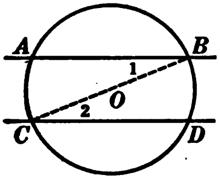


FIG. 1.

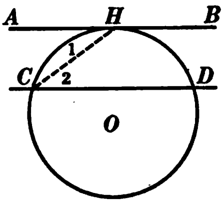


FIG. 2.

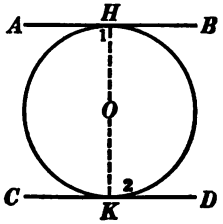


FIG. 3.

I. If the two ∥ lines are secants or chords (Fig. 1).

Given ∥ chords *AB* and *CD*, intercepting arcs *AC* and *BD*.

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

ARGUMENT	REASONS
1. Draw *CB*.	1. § 54, 15.
2. $\angle 1 = \angle 2$.	2. § 189.
3. $\angle 1 \overset{\propto}{=} \frac{1}{2}\overset{\frown}{AC}$, and $\angle 2 \overset{\propto}{=} \frac{1}{2}\overset{\frown}{BD}$.	3. § 365.
4. $\therefore \frac{1}{2}\overset{\frown}{AC} = \frac{1}{2}\overset{\frown}{BD}$.	4. § 362, *a*.
5. $\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$. Q.E.D.	5. § 54, 7 *a*.

II. If one of the ∥ lines is a secant and the other a tangent (Fig. 2).

III. If the two ∥ lines are tangents (Fig. 3).

The proofs of II and III are left as exercises for the student.

Ex. 512. In Fig. 1, Prop. XXV, if angle 1 equals 42° and arc *DE* equals 40°, how many degrees are there in angle 3? in arc *AC*?

Ex. 513. In Fig. 2, if arc *DC* equals 60° and angle 3 equals 100°, find the number of degrees in angle 1.

Ex. 514. In Fig. 3, if angle 1 equals 65°, find angle 2 and angle 3.

Ex. 515. An angle formed by two tangents is measured by 180° minus the intercepted arc.

Ex. 516. If two tangents to a circle meet at an angle of 40°, how many degrees of arc do they intercept?

Ex. 517. Is the converse of Prop. XXVI true? Prove your answer.

Ex. 518. If two sides of an inscribed quadrilateral are parallel, the other two sides are equal.

Ex. 519. Is the converse of Ex. 518 true? Prove your answer.

Ex. 520. If, through the point where the bisector of an inscribed angle cuts the circumference, a chord is drawn parallel to one side of the angle, this chord will equal the other side of the angle.

Ex. 521. If through the points of intersection of two circumferences parallels are drawn terminating in the circumferences, these parallels will be equal.

Ex. 522. Prove that a trapezoid inscribed in a circle is isosceles.

Ex. 523. If two sides of an inscribed hexagon are equal and parallel, and two other sides are parallel, the remaining sides are equal and parallel.

Ex. 524. If the sides AB and BC of an inscribed quadrilateral $ABCD$ subtend arcs of 60° and 130°, respectively, and if angle AED, formed by the diagonals AC and BD intersecting at E, is 75°, how many degrees are there in arcs AD and DC? how many degrees in each angle of the quadrilateral?

MISCELLANEOUS EXERCISES

Ex. 525. Equal chords of a circle whose center is C intersect at E. Prove that CE bisects the angle formed by the chords.

Ex. 526. A common tangent to two unequal circles intersects their line of centers at a point P; from P a second tangent is drawn to one of the circles. Prove that it is also tangent to the other.

Ex. 527. Through one of the points of intersection of two circumferences draw a chord of one that shall be bisected by the other circumference.

Ex. 528. The angle ABC is any inscribed angle in a given segment of a circle; AC is prolonged to P, making CP equal to CB. Find the locus of P.

Ex. 529. Given two points P and Q, and a straight line through Q. Find the locus of the foot of the perpendicular from P to the given line, as the latter revolves around Q.

Ex. 530. If two circles touch each other and a line is drawn through the point of contact and terminated by the circumferences, the tangents at its ends are parallel.

Ex. 531. If two circles touch each other and two lines are drawn through the point of contact terminated by the circumferences, the chords joining the ends of these lines are parallel.

Ex. 532. If one arm of a right triangle is the diameter of a circle, the tangent at the point where the circumference cuts the hypotenuse bisects the other arm.

Ex. 533. Two fixed circles touch each other externally and a circle of variable radius touches both externally. Show that the difference of the distances from the center of the variable circle to the centers of the fixed circles is constant.

Ex. 534. If two circles are tangent externally, their common internal tangent bisects their common external tangent.

Ex. 535. If two circles are tangent externally and if their common external tangent is drawn, lines drawn from the point of contact of the circles to the points of contact of the external tangent are perpendicular to each other.

Ex. 536. The two common external tangents to two circles meet their line of centers at a common point. Also the two common internal tangents meet the line of centers at a common point.

Ex. 537. Two circles whose radii are 17 and 10 inches, respectively, are tangent externally. How long is the line joining their centers? how long if the same circles are tangent internally?

Ex. 538. If a right angle at the center of a circle is trisected, is the intercepted arc also trisected? Is the chord which subtends the arc trisected?

Ex. 539. Draw a line intersecting two given circumferences in such a way that the chords intercepted by the two circumferences shall equal two given lines. What restriction is there on the lengths of the given lines?

Ex. 540. Construct a triangle, given its base, the vertex angle, and the median to the base. Under what conditions will there be no solution?

Ex. 541. In the same circle, or in equal circles, two inscribed triangles are equal, if two sides of one are equal respectively to two sides of the other.

Ex. 542. If through the points of intersection of two circumferences two lines are drawn terminating in the circumferences, the chords which join their extremities are parallel.

Ex. 543. The tangents drawn through the vertices of an inscribed rectangle, which is not a square, form a rhombus.

Ex. 544. The line joining the center of the square described upon the hypotenuse of a right triangle, to the vertex of the right angle, bisects the right angle.

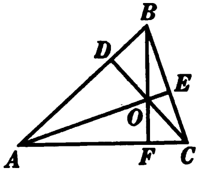
Ex. 545. If upon the sides of any triangle equilateral triangles are drawn, and circles circumscribed about the three triangles, these circles will intersect at a common point.

Ex. 546. Through two given points draw two parallel lines at a given distance apart.

Ex. 547. In a given circle inscribe a chord of given length which prolonged shall be tangent to another given circle.

Ex. 548. Find the locus of the middle point of a chord drawn from a given point in a given circumference.

Ex. 549. The locus of the intersections of the altitudes of triangles having a given base and a given angle at the vertex is the arc which forms with the base a segment capable of containing an angle equal to the supplement of the given angle at the vertex.



HINT. Let ABC be one of the △ and O the intersection of the altitudes. In quadrilateral $FOEC$, $\angle ECF$ is the supplement of $\angle FOE$. $\therefore \angle ECF$ is the supplement of $\angle BOA$.

Ex. 550. In a circle, prove that any chord which bisects a radius at right angles subtends an angle of 120° at the center.

Ex. 551. Construct an equilateral triangle, having given the radius of the inscribed circle.

Ex. 552. The two circles described upon two sides of a triangle as diameters intersect upon the third side.

Ex. 553. All triangles circumscribed about the same circle and mutually equiangular are equal.

Ex. 554. If two circumferences meet on their line of centers, the circles are tangent to each other.

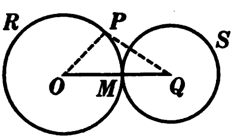


FIG. 1.

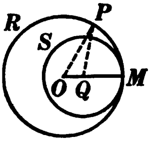


FIG. 2.

OUTLINE OF PROOF

I. M between O and Q, Fig. 1.

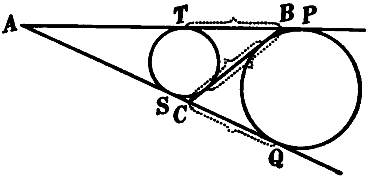
$OP + PQ > OQ$.
$OP = OM$.
$\therefore PQ > MQ$.
$\therefore$ P is not on circumference S.

II. M not between O and Q, Fig. 2.

$OQ + QP > OP$.
$\therefore OQ + QP > OM$.
$\therefore QP > QM$.
$\therefore$ P is not on circumference S.

Ex. 555. If two circumferences intersect, neither point of intersection is on their line of centers.

Ex. 556. In any right triangle ABC, right-angled at C, the radius of the inscribed circle equals $\frac{1}{2}(a + b - c)$ and the radius of the escribed circle tangent to c equals $\frac{1}{2}(a + b + c)$.



Ex. 557. In the accompanying figure a, b, c, are the sides of triangle ABC. Prove:

1. $a = TP$;
2. $AP = \frac{1}{2}(a + b + c)$;
3. $TB = \frac{1}{2}(a + c - b)$.

Ex. 558. Trisect a quadrant; a semicircumference; a circumference.

Ex. 559. Describe circles about the vertices of a given triangle as centers, so that each shall touch the other two.

Ex. 560. Construct within a given circle three equal circles, so that each shall touch the other two and also the given circle.

Construct a triangle, having given:

Ex. 561. h_a, h_c, C.

Ex. 562. A, B, and R, the radius of the circumscribed circle.

Ex. 563. a, B, R.

Ex. 564. C and the segments of c made by t_c.

Ex. 565. C and the segments of c made by h_c.

Ex. 566. R, the radius of the circumscribed circle, and the segments of c made by t_c.

Construct a right triangle ABC, right-angled at C, having given:

Ex. 567. c, h_c.

Ex. 568. c and one segment of c made by h_c.

Ex. 569. The segments of c made by h_c.

Ex. 570. The segments of c made by t_c.

Ex. 571. c and a line l, in which the vertex of C must lie.

Ex. 572. c and the perpendicular from vertex C to a line l.

Ex. 573. c and the distance from C to a point P.

Construct a square, having given;

Ex. 574. The perimeter.

Ex. 575. A diagonal.

Ex. 576. The sum of a diagonal and a side.

Construct a rectangle, having given:

Ex. 577. Two non-parallel sides.

Ex. 578. A side and a diagonal.

Ex. 579. The perimeter and a diagonal.

Ex. 580. A diagonal and an angle formed by the diagonals.

Ex. 581. A side and an angle formed by the diagonals.

Ex. 582. The perimeter and an angle formed by the diagonals.

Construct a rhombus, having given:

Ex. 583. A side and a diagonal.

Ex. 584. The perimeter and a diagonal.

Ex. 585. One angle and a diagonal.

Ex. 586. The two diagonals.

Ex. 587. A side and the sum of the diagonals.

Ex. 588. A side and the difference of the diagonals.

Construct a parallelogram, having given:

Ex. 589. Two non-parallel sides and an angle.

Ex. 590. Two non-parallel sides and a diagonal.

Ex. 591. One side and the two diagonals.

Ex. 592. The diagonals and an angle formed by them.

Construct an isosceles trapezoid, having given:

Ex. 593. The bases and a diagonal.

Ex. 594. The longer base, the altitude, and one of the equal sides.

Ex. 595. The shorter base, the altitude, and one of the equal sides

Ex. 596. Two sides and their included angle.

Construct a trapezoid, having given:

Ex. 597. The bases and the angles adjacent to one base.

Ex. 598. The bases, the altitude, and an angle.

Ex. 599. One base, the two diagonals, and their included angle.

Ex. 600. The bases, a diagonal, and the angle between the diagonals

Construct a circle which shall:

Ex. 601. Touch a given circle at P and pass through a given point Q

Ex. 602. Touch a given line l at P.

Ex. 603. Touch three given lines two of which are parallel.

Ex. 604. Touch a given line l at P and also touch another line m.

Ex. 605. Have its center in line l, cut l at P, and touch a circle K

BOOK III

PROPORTION AND SIMILAR FIGURES

382. Def. A **proportion** is the expression of the equality of two ratios.

EXAMPLE. If the ratio $\frac{a}{b}$ is equal to the ratio $\frac{c}{d}$, then the equation $\frac{a}{b}=\frac{c}{d}$ is a proportion. This proportion may also be written $a : b = c : d$ or $a : b :: c : d$, and is read *a is to b as c is to d*.

383. Def. The four numbers a, b, c, d are called the **terms** of the proportion.

384. Def. The first term of a *ratio* is called its **antecedent** and the second term its **consequent**; therefore:

The first and third terms of a *proportion* are called **antecedents**, and the second and fourth terms, **consequents.**

385. Def. The second and third terms of a proportion are called its **means**, and the first and fourth terms, its **extremes.**

386. Def. If the two means of a proportion are equal, this common mean is called the **mean proportional** between the two extremes, and the last term of the proportion is called the **third proportional** to the first and second terms *taken in order;* thus, in the proportion $a : b = b : c$, b is the mean proportional between a and c, and c is the third proportional to a and b.

387. Def. The **fourth proportional** to three given numbers is the fourth term of a proportion the first three terms of which are the three given numbers *taken in order;* thus, if $a : b = c : d$, d is called the fourth proportional to a, b, and c.

PROPOSITION I. THEOREM

388. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Given $a : b = c : d$.

To prove $ad = bc$.

	ARGUMENT		REASONS
1.	$\frac{a}{b} = \frac{c}{d}$.		1. By hyp.[*]
2.	$bd = bd$.		2. By iden.
3.	$\therefore ad = bc$.	Q.E.D.	3. § 54, 7*a*.

389. Note. The student should observe that the process used here is merely the algebraic "clearing of fractions," and that as fractional equations in algebra are usually simplified by this process, so, also, proportions may be simplified by placing the product of the means equal to the product of the extremes.

390. Cor. I. *The mean proportional between two numbers is equal to the square root of their product.*

391. Cor. II. *If two proportions have any three terms of one equal respectively to the three corresponding terms of the other, then the remaining term of the first is equal to the remaining term of the second.*

Ex. 606. Given the equation $m : r = d : c$; solve (1) for d, (2) for r, (3) for m, (4) for c.

Ex. 607. Find the fourth proportional to 4, 6, and 10; to 4, 10, and 6, to 10, 6, and 4.

Ex. 608. Find the mean proportional between 9 and 144; between 144 and 9.

Ex. 609. Find the third proportional to $\frac{4}{5}$ and $\frac{2}{3}$; to $\frac{2}{3}$ and $\frac{4}{5}$.

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392. Questions. What rearrangement of numbers can be made in Ex. 607 without affecting the required term? in Ex. 608? in Ex. 609?

Ex. 610. Find the third proportional to $a^2 - b^2$ and $a - b$.

Ex. 611. Find the fourth proportional to $a^2 - b^2$, $a - b$, and $a + b$

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[*] See § 382 for the three ways of writing a proportion.

Ex. 612. If in any proportion the antecedents are equal, then the consequents are equal and conversely.

Ex. 613. If $a : b = c : d$, prove that $ma : kb = mc : kd$.

Ex. 614. If $l : k = b : m$, prove that $lr : kr = bc : mc$.

Ex. 615. If $x : y = b : c$, prove that $dx : y = bd : c$.

Ex. 616. If $x : y = b : c$, is $dx : y = b : cd$ a true proportion?

PROPOSITION II. THEOREM

(Converse of Prop. I)

393 *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means and the other pair the extremes of a proportion.*

Given $ad = bc$.

To prove $a : b = c : d$.

ARGUMENT	REASONS
1. $ad = bc$.	1. By hyp.
2. $bd = bd$.	2. By iden.
3. $\therefore \frac{a}{b} = \frac{c}{d}$; i.e. $a : b = c : d$. Q.E.D.	3. § 54, 8*a*.

The proof that a and d may be made the means and b and c the extremes is left as an exercise for the student.

394. Note. The pupil should observe that the divisor in Arg. 2 above must be chosen so as to give the desired quotient in the first member of the equation: thus, if $hl = kf$, and we wish to prove that $\frac{h}{f} = \frac{k}{l}$, we must divide by fl; then $\frac{hl}{fl} = \frac{kf}{fl}$, i.e. $\frac{h}{f} = \frac{k}{l}$.

Ex. 617. Given $pt = cr$. Prove $p : r = c : t$; also, $c : p = t : r$.

Ex. 618. From the equation $rs = lm$, derive the following eight proportions:

$r : l = m : s$, $s : l = m : r$, $l : r = s : m$, $m : r = s : l$

$r : m = l : s$, $s : m = l : r$, $l : s = r : m$, $m : s = r : l$.

Ex. 619. Form a proportion from $7 \times 4 = 3 \times a$; from $ft = gb$. How can the proportions obtained be verified?

Ex. 620. Form a proportion from $(a+c)(a-b)=de$.

Ex. 621. Form a proportion from $m^2-2mn+n^2=ab$.

Ex. 622. Form a proportion from $c^2+2cd+d^2=a+b$.

Ex. 623. Form a proportion from $(a+b)(a-b)=4x$, making x (1) an extreme; (2) a mean.

Ex. 624. If $7x+8y:12=2x+y:3$, find the ratio $x:y$.

PROPOSITION III. THEOREM

395. *If four numbers are in proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth is to the third.*

Given $a:b=c:d$.

To prove $b:a=d:c$.

	ARGUMENT		REASONS
1.	$a:b=c:d$.		1. By hyp.
2.	$\therefore ad=bc$.		2. § 388.
3.	$\therefore b:a=d:c$.	Q.E.D.	3. § 393.

PROPOSITION IV. THEOREM

396. *If four numbers are in proportion, they are in proportion by alternation; that is, the first term is to the third as the second is to the fourth.*

Given $a:b=c:d$.

To prove $a:c=b:d$.

	ARGUMENT		REASONS
1.	$a:b=c:d$.		1. By hyp.
2.	$\therefore ad=bc$.		2. § 388.
3.	$\therefore a:c=b:d$.	Q.E.D.	3. § 393.

Ex. 625. If $x:\frac{14}{3}=y:\frac{7}{3}$, what is the value of the ratio $x:y$?

Ex. 626. Transform $a:x=4:3$ so that x shall occupy in turn every place in the proportion.

397. Many transformations may be easily brought about by the following method:

(1) Reduce the conclusion to an equation in its simplest form (§ 388), then from this derive the hypothesis.

(2) Begin with the hypothesis and reverse the steps of (1).

This method is illustrated in the analysis of Prop. V.

PROPOSITION V. THEOREM

398. *If four numbers are in proportion, they are in proportion by composition; that is, the sum of the first two terms is to the first (or second) term as the sum of the last two terms is to the third (or fourth) term.*

Given $a:b=c:d$.

To prove: (*a*) $a+b:a=c+d:c$;
(*b*) $a+b:b=c+d:d$.

I. Analysis

(1) The conclusions required above, when reduced to equations in their simplest forms, are as follows:

(*a*)	(*b*)
1. $ac+ad=ac+bc$.	1. $bc+bd=ad+bd$.
2. Whence $ad=bc$.	2. Whence $bc=ad$.
3. $\therefore a:b=c:d$.	3. $\therefore a:b=c:d$.

(2) Now begin with the hypothesis and reverse the steps.

II. Proof

(*a*)

	ARGUMENT	REASONS
1.	$a:b=c:d$.	1. By hyp.
2.	$\therefore ad=bc$.	2. § 388.
3.	$ac=ac$.	3. By iden.
4.	$\therefore ac+ad=ac+bc$; i.e. $a(c+d)=c(a+b)$.	4. § 54, 2.
5.	$\therefore a+b:a=c+d:c$. Q.E.D.	5. § 393.

(*b*) The proof of (*b*) is left as an exercise for the student.

Ex. 627. If $\frac{x}{y} = \frac{6}{5}$, find $\frac{x+y}{x}$; $\frac{x+y}{y}$.

PROPOSITION VI. THEOREM

399. *If four numbers are in proportion, they are in proportion by* ***division;*** *that is, the difference of the first two terms is to the first (or second) term as the difference of the last two terms is to the third (or fourth) term.*

Given $a : b = c : d$.

To prove: (*a*) $a - b : a = c - d : c$;
(*b*) $a - b : b = c - d : d$.

I. The analysis is left as an exercise for the student.

II. Proof

(*a*)	ARGUMENT	REASONS
1.	$a : b = c : d$.	1. By hyp.
2.	$\therefore ad = bc$.	2. § 388.
3.	$ac = ac$.	3. By iden.
4.	$\therefore ac - ad = ac - bc$, i.e. $a(c - d) = c(a - b)$.	4. § 54, 3.
5.	$\therefore a - b : a = c - d : c$. Q.E.D.	5. § 393.

(*b*) The proof of (*b*) is left as an exercise for the student.

Ex. 628. If $\frac{x}{y} = \frac{8}{3}$, find $\frac{x-y}{x}$; $\frac{x-y}{y}$.

PROPOSITION VII. THEOREM

400. *If four numbers are in proportion, they are in proportion by* ***composition and division;*** *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given $a : b = c : d$.

To prove $a + b : a - b = c + d : c - d$.

Argument	Reasons
1. $a:b=c:d.$	1. By hyp.
2. $\therefore \frac{a+b}{a}=\frac{c+d}{c}.$	2. § 398.
3. And $\frac{a-b}{a}=\frac{c-d}{c}.$	3. § 399.
4. $\therefore \frac{a+b}{a-b}=\frac{c+d}{c-d};$	4. § 54, 8 a.
i.e. $a+b:a-b=c+d:c-d.$ Q.E.D.	

PROPOSITION VIII. THEOREM

401. *In a series of equal ratios the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Given $a:b=c:d=e:f=g:h.$

To prove $a+c+e+g:b+d+f+h=a:b.$

I. Analysis

Simplifying the conclusion above, we have:

$$ab+bc+be+bg=ab+ad+af+ah.$$

The terms required for the first member of this equation, and their equivalents for the second member, may be obtained from the hypothesis.

II. Proof

Argument	Reasons
1. $ab=ab.$	1. By iden.
2. $bc=ad.$	2. § 388.
3. $be=af.$	3. § 388.
4. $bg=ah.$	4. § 388.
5. $\therefore ab+bc+be+bg=ab+ad+af+ah;$ i.e. $b(a+c+e+g)=a(b+d+f+h).$	5. § 54, 2.
6. $\therefore a+c+e+g:b+d+f+h=a:b.$ Q.E.D.	6. § 393.

Ex. 629. With the hypothesis of Prop. VIII, prove

$$a + c + e : b + d + f = g : h.$$

Ex. 630. If $\frac{m}{r} = \frac{s}{t} = \frac{k}{g} = \frac{p}{q}$, prove $\frac{m - s + k - p}{r - t + g - q} = \frac{p}{q}$.

Ex. 631. If $\frac{x + 2y}{2y} = \frac{7}{4}$, find $\frac{x}{y}$. Hint. Use Prop. VI.

Ex. 632. If $a : b = c : d$, show that $b - a : b + a = d - c : d + c$.

Ex. 633. Given the proportion $a : b = 11 : 6$. Write the proportions that result from taking the terms (1) by inversion; (2) by alternation; (3) by composition; (4) by division; (5) by composition and division.

Proposition IX. Theorem

402. *The products of the corresponding terms of any number of proportions form a proportion.*

Given $a : b = c : d,\ e : f = g : h,\ i : j = k : l.$

To prove $aei : bfj = cgk : dhl.$

	Argument	Reasons
1.	$\frac{a}{b} = \frac{c}{d}$; $\frac{e}{f} = \frac{g}{h}$; $\frac{i}{j} = \frac{k}{l}$.	1. By hyp.
2.	$\therefore \frac{aei}{bfj} = \frac{cgk}{dhl}$; i.e. $aei : bfj = cgk : dhl$. Q.E.D.	2. § 54, 7 a.

403. Cor. *If four numbers are in proportion, equimultiples of the first two and equimultiples of the last two are also in proportion.*

Hint. **Given** $a : b = x : y$.
To prove $am : bm = nx : ny$.

Ex. 634. If $r : s = m : t$, is $fr : qs = qm : ft$? Prove your answer.

Ex. 635. If $a : b = 3 : 4$ and $x : y = 8 : 9$, find the value of $ax : by$

Ex. 636. If four numbers are in proportion, equimultiples of the antecedents and equimultiples of the consequents are also in proportion.

Ex. 637. If $r : s = t : m$, prove $3s + 2m : 4s = 3r + 2t : 4r$.

Ex. 638. If $w : x = y : z$, prove $ax + bz : cx + dz = aw + by : cw + dy$.

Ex. 639. If $\frac{m}{r} = \frac{k}{v}$ and $\frac{n}{s} = \frac{l}{w}$, prove that $\frac{m}{n} : \frac{r}{s} = \frac{k}{l} : \frac{v}{w}$, and state the theorem thus derived.

PROPOSITION X. THEOREM

404. *If four numbers are in proportion, like powers of these numbers are in proportion, and so also are like roots.*

Given $a : b = c : d$.

To prove: (*a*) $a^p : b^p = c^p : d^p$.

(*b*) $\sqrt[r]{a} : \sqrt[r]{b} = \sqrt[r]{c} : \sqrt[r]{d}$.

ARGUMENT	REASONS
1. $\frac{a}{b} = \frac{c}{d}$.	1. By hyp.
2. $\therefore \frac{a^p}{b^p} = \frac{c^p}{d^p}$; i.e. $a^p : b^p = c^p : d^p$.	2. § 54, 13.
3. Also $\frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \frac{\sqrt[r]{c}}{\sqrt[r]{d}}$; i.e. $\sqrt[r]{a} : \sqrt[r]{b} = \sqrt[r]{c} : \sqrt[r]{d}$.	3. § 54, 13.
Q.E.D.	

Ex. 640. If $\frac{r}{s} = \frac{p}{q}$, is $\frac{\sqrt{r}}{\sqrt{s}} = \frac{p}{q}$? is $\frac{3r}{3s} = \frac{p}{q}$?

Ex. 641. If $\frac{m}{n} = \frac{p}{q} = \frac{r}{s}$, prove that $\frac{m + 2p}{n + 2q} = \frac{p + 3r}{q + 3s} = \frac{r + 4m}{s + 4n}$.

405. Def. A **continued proportion** is a series of equal ratios in which the consequent of *any* ratio is the same number as the antecedent of the following ratio; thus,

$a : b = c : d = e : f = g : h$ is merely a series of equal ratios, while $a : b = b : c = c : d = d : e$ is not only a series of equal ratios but a continued proportion as well.

Ex. 642. In the continued proportion $a:b=b:c=c:d=d:e$, prove that:

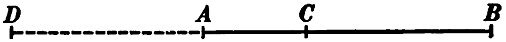
$$\frac{a}{c}=\frac{a^2}{b^2};\ \frac{a}{d}=\frac{a^3}{b^3};\ \text{and}\ \frac{a}{e}=\frac{a^4}{b^4}.$$

Ex. 643. If $x^3:y^3=8:27$, find $\frac{x}{y}$.

Ex. 644. If $\sqrt{m}:1=\sqrt{n}:16$, find $\frac{m}{n}$.

---

406. Def. The **segments of a line** are the parts into which it is divided. The line *AB* is **divided internally** at *C* if this point is between the extremities of the line. The segments into which it is divided are *AC* and *CB*.



AB is **divided externally** at *D* if this point is on the prolongation of the line. The segments are *AD* and *DB*.

It should be noted that in either case *the point of division is one end of each segment.*

407. Def. Two straight lines are **divided proportionally** if the ratio of one line to either of its segments is equal to the ratio of the other line to its corresponding segment.

408. In Prop. XI, II, the following theorems (Appendix §§ 586 and 591) will be assumed:

(*a*) *The quotient of a variable by a constant is a variable.*

(*b*) *The limit of the quotient of a variable by a constant is the limit of the variable divided by the constant.*

Thus, if x is a variable and k a constant:

(1) $\frac{x}{k}$ is a variable.

(2) If the limit of x is y, then the limit of $\frac{x}{k}$ is $\frac{y}{k}$.

Ex. 645. In the figure of § 409, name the segments into which *AB* is divided by *D*; the segments into which *AD* is divided by *B*.

Proposition XI. Theorem

409. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*

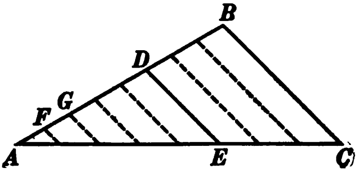


FIG. 1.

Given $\triangle ABC$ with line $DE \parallel BC$.

To prove $\frac{AB}{AD} = \frac{AC}{AE}$.

I. If AB and AD are commensurable (Fig. 1).

Argument	Reasons
1. Let AF be a common measure of AB and AD, and suppose that AF is contained in AB r times and in AD s times.	1. § 335.
2. Then $\frac{AB}{AD} = \frac{r}{s}$.	2. § 341.
3. Through the several points of division on AB, as F, G, etc., draw lines $\parallel BC$.	3. § 179.
4. These lines are $\parallel DE$ and to each other.	4. § 180.
5. $\therefore$ AC is divided into r equal parts and AE into s equal parts.	5. § 244.
6. $\therefore \frac{AC}{AE} = \frac{r}{s}$.	6. § 341.
7. $\therefore \frac{AB}{AD} = \frac{AC}{AE}$. Q.E.D.	7. § 54, 1.

II. If AB and AD are incommensurable (Fig. 2).

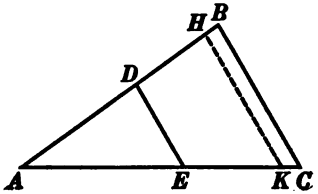


Fig. 2.

Argument	Reasons
1 Let m be a measure of AD. Apply m as a measure to AB as many times as possible. There will then be a remainder, HB, less than m.	1. § 339.
2. AH and AD are commensurable.	2. § 337.
3. Draw $HK \parallel BC$.	3. § 179.
4. $\therefore \frac{AH}{AD} = \frac{AK}{AE}$.	4. § 409, I.
5. Now take a smaller measure of AD. No matter how small a measure of AD is taken, when it is applied as a measure to AB, the remainder, HB, will be smaller than the measure taken.	5. § 335.
6. $\therefore$ the difference between AH and AB may be made to become and remain less than any previously assigned line, however small.	6. Arg. 5.
7. $\therefore$ AH approaches AB as a limit.	7. § 349.
8. $\therefore \frac{AH}{AD}$ approaches $\frac{AB}{AD}$ as a limit.	8. § 408, b.
9. Likewise the difference between AK and AC may be made to become and remain less than any previously assigned line, however small.	9. Arg. 6.

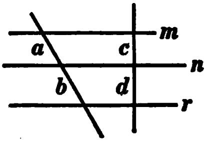
Argument	Reasons
10. $\therefore AK$ approaches AC as a limit.	10. § 349.
11. $\therefore \frac{AK}{AE}$ approaches $\frac{AC}{AE}$ as a limit.	11. § 408, *b*.
12. But $\frac{AH}{AD}$ is always equal to $\frac{AK}{AE}$.	12. Arg. 4.
13. $\therefore \frac{AB}{AD} = \frac{AC}{AE}$. Q.E.D.	13. § 355.

410. Cor. *A straight line parallel to one side of a triangle divides the other two sides into segments which are proportional.* Thus, in the figures for Prop. XI, $AD:DB = AE:EC$.

HINT. Prove by using division.

Ex. 646. Using Fig. 2 of Prop. XI, prove

(1) $AB:AC = AD:AE$; (2) $AB:AC = DB:EC$.



Ex. 647. In the diagram at the right, m, n, and r are parallel to each other. Prove that $a:b = c:d$; also that $a:c = b:d$.

Ex. 648. If two sides of a triangle are 12 inches and 18 inches, and if a line is drawn parallel to the third side and cuts off 3 inches from the vertex on the 12-inch side, into what segments will it cut the 18-inch side?

Ex. 649. If two lines are cut by any number of parallels, the two lines are divided into segments which are proportional: (a) if the two lines are parallel; (b) if the two lines are oblique.

Ex. 650. If through the point of intersection of the medians of a triangle a line is drawn parallel to any side of the triangle, this line divides the other two sides in the ratio of 2 to 1.

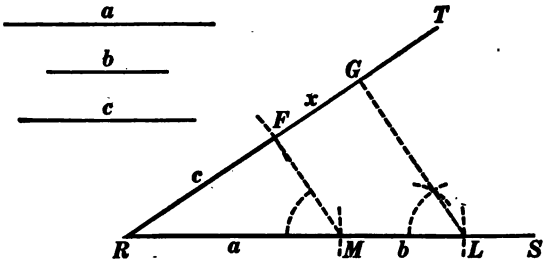
Ex. 651. A line can be divided at but one point into segments which have a given ratio (measured from one end).

Ex. 652. A line parallel to the bases of a trapezoid divides the other two sides and also the two diagonals proportionally.

Ex. 653. Apply the proof of Prop. XI to the case in which the parallel to the base cuts the sides prolonged: (a) through the ends of the base; (b) through the vertex.

PROPOSITION XII. PROBLEM

411. *To construct the fourth proportional to three given lines.*



Given lines a, b, and c.

To construct the fourth proportional to a, b, and c.

I. Construction

1. From any point, as R, draw two indefinite lines RS and RT.
2. On RS lay off $RM = a$ and $ML = b$.
3. On RT lay off $RF = c$.
4. Draw MF.
5. Through L construct $LG \parallel MF$. § 188.
6. FG is the fourth proportional to a, b, and c.

II. The proof and discussion are left as an exercise for the student.

412. Question. Could the segments a, b, and c be laid off in any other order?

413. Cor. I. *To construct the third proportional to two given lines.*

414. Cor II. *To divide a given line into segments proportional to two or more given lines.*

Ex. 654. Divide a given line into segments in the ratio of 3 to 5.

Ex. 655. Divide a given line into segments proportional to 2, 3, and 4.

Ex. 656. Construct two lines, given their sum and their ratio.

Ex. 657. Construct two lines, given their difference and their ratio.

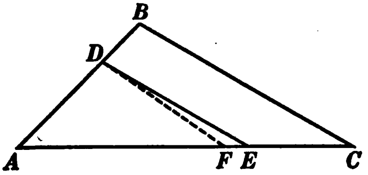
Ex. 658. If a, b, and c are three given lines, construct x so that:

$$(a)\ x : a = b : c;\ (b)\ x = \frac{ac}{b}.$$

PROPOSITION XIII. THEOREM

(Converse of Prop. XI)

415. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

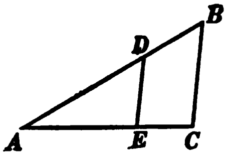


Given $\triangle ABC$, and DE so drawn that $\frac{AB}{AD} = \frac{AC}{AE}$.

To prove $DE \parallel BC$.

ARGUMENT	REASONS
1. DE and BC are either $\parallel$ or not $\parallel$.	1. § 161, *a*.
2. Suppose that DE is not $\parallel BC$, but that some other line through D, as DF, is $\parallel BC$.	2. § 179.
3. Then $\frac{AB}{AD} = \frac{AC}{AF}$.	3. § 409.
4. But $\frac{AB}{AD} = \frac{AC}{AE}$.	4. By hyp.
5. $\therefore\ AF = AE$.	5. § 391.
6. This is impossible.	6. § 54, 12.
7. $\therefore DE \parallel BC$. Q.E.D.	7. § 161, *b*.

416. Cor. *If a straight line divides two sides of a triangle into segments which are proportional, it is parallel to the third side.* Thus, if $AD : DB = AE : EC$, DE is $\parallel BC$.



Ex. 659. In the diagram at the right, if $AB = 15$, $AC = 12$, $AD = 10$, and $AE = 8$, prove DE parallel to BC.

Ex. 660. If $AB = 50$, $DB = 15$, $AE = 28$, and $EC = 12$, is DE parallel to BC? Prove.

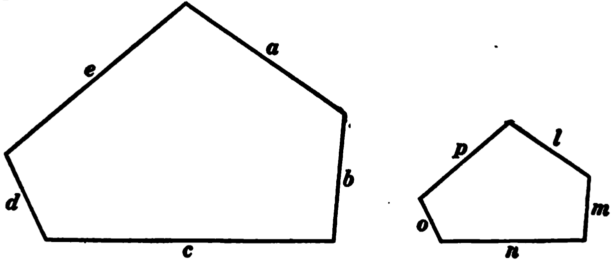
Ex. 661. If $DE \parallel BC$, $AB = 25$, $DB = 5$, and $AC = 20$, find AE.

Ex. 662. If $DE \parallel BC$, $AB = 30$, $AD = 25$, and $EC = 4$, find AC.

SIMILAR POLYGONS

417. Def. If the angles of one polygon, taken in order, are equal respectively to those of another, taken in order, the polygons are said to be **mutually equiangular.** The pairs of equal angles in the two polygons, taken in order, are called **homologous angles** of the two polygons.

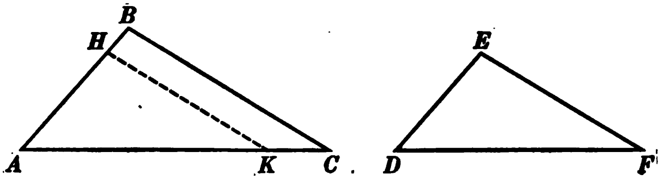
418. Def. If the sides of one polygon, taken in order as antecedents, form a series of equal ratios with the sides of another polygon, taken in order as consequents, the polygons are said to have their **sides proportional.** Thus, in the accompanying figure, if $a : l = b : m = c : n = d : o = e : p$, the two polygons have their sides proportional. The lines forming any ratio are called **homologous lines** of the two polygons, and the ratio of two such lines is called the **ratio of similitude** of the polygons.



419. Def. Two **polygons** are **similar** if they are mutually equiangular and if their sides are proportional.

Proposition XIV. Theorem

420. *Two triangles which are mutually equiangular are similar.*



Given $\triangle s$ ABC and DEF with $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

To prove $\triangle ABC \sim \triangle DEF$.

Argument	Reasons
1. Place $\triangle DEF$ on $\triangle ABC$ so that $\angle D$ shall coincide with $\angle A$, DE falling on AB and DF on AC. Represent $\triangle DEF$ in its new position by $\triangle AHK$.	1. § 54, 14.
2. $\angle B = \angle E = \angle AHK$.	2. By hyp.
3. $\therefore HK \parallel BC$.	3. § 184.
4. $\therefore \frac{AB}{AH} = \frac{AC}{AK}$.	4. § 409.
5. $\therefore \frac{AB}{DE} = \frac{AC}{DF}$.	5. § 309.
6. By placing $\triangle DEF$ on $\triangle ABC$ so that $\angle E$ shall coincide with $\angle B$, it may be shown that $\frac{AB}{DE} = \frac{BC}{EF}$.	6. By steps similar to 1–5.
7. $\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.	7. § 54, 1.
8. $\therefore \triangle ABC \sim \triangle DEF$. Q.E.D.	8. § 419.

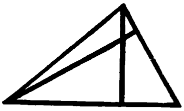
421. Cor. I. *If two triangles have two angles of one equal respectively to two angles of the other, the triangles are similar.*

422. Cor. II. *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

423. Cor. III. *If a line is drawn parallel to any side of a triangle, this line, with the other two sides, forms a triangle which is similar to the given triangle.*

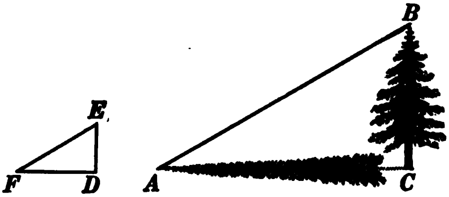
Ex. 663. Upon a given line as base construct a triangle similar to a given triangle.

Ex. 664. Draw a triangle *ABC*. Estimate the lengths of its sides. Draw a second triangle *DEF* similar to *ABC* and having *DE* equal to two thirds of *AB*. Compute *DF* and *EF*.



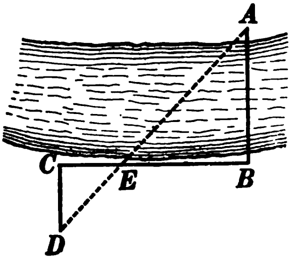
Ex. 665. Any two altitudes of a triangle are to each other inversely as the sides to which they are drawn.

Ex. 666. At a certain hour of the day a tree, *BC*, casts a shadow, *CA*. At the same time a vertical pole, *ED*, casts a shadow, *DF*. What measurements are necessary to determine the height of the tree?



Ex. 667. If *CA* is found to be 64 feet; *DF*, 16 feet; *ED*, 10 feet what is the height of the tree?

Ex. 668. To find the distance across a river from *A* to *B*, a point *C* was located so that *BC* was perpendicular to *AB* at *B*. *CD* was then measured off 100 feet in length and perpendicular to *BC* at *C*. The line of sight from *D* to *A* intersected *BC* at *E*. By measurement *CE* was found to be 90 feet and *EB* 210 feet. What was the distance across the river?



Ex. 669. Two isosceles triangles are similar if the vertex angle of one equals the vertex angle of the other, or if a base angle of one equals a base angle of the other.

424. It follows from the definition of similar polygons, § 419, and from Prop. XIV that:

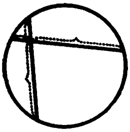
(1) *Homologous angles of similar triangles are equal.*

(2) *Homologous sides of similar triangles are proportional.*

(3) *Homologous sides of similar triangles are the sides opposite equal angles.*

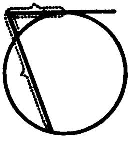
425. Note. In case, therefore, it is desired to prove four lines proportional, try to find a pair of triangles each having two of the given lines as sides. If, then, these triangles can be proved similar, their homologous sides will be proportional. By *marking with colored crayon* the lines required in the proportion, the triangles can readily be found. If it is desired to prove the product of two lines equal to the product of two other lines, prove the four lines proportional by the method just suggested, then put the product of the extremes equal to the product of the means.*

426. Def. **The length of a secant** from an external point to a circle is the length of the segment included between the point and the second point of intersection of the secant and the circumference.



Ex. 670. If two chords intersect within a circle, establish a proportionality among the segments of the chords. Place the product of the extremes equal to the product of the means, and state your result as a theorem.

Ex. 671. If two secants are drawn from any given point to a circle, what are the segments of the secants? Does the theorem of Ex. 670 still hold with regard to them?



Ex. 672. Rotate one of the secants of Ex. 671 about the point of intersection of the two until the rotating secant becomes a tangent. What are the segments of the secant which has become a tangent? Does the theorem of Ex 670 still hold? Prove.

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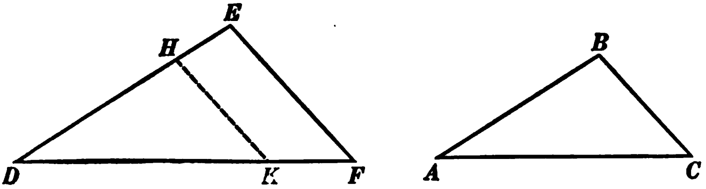
* By the product of two lines is meant the product of their measure-numbers. This will be discussed again in Book IV.

PROPOSITION XV. THEOREM

427. ***Two triangles which have their sides proportional are similar.***



Given $\triangle$s DEF and ABC such that $\frac{DE}{AB}=\frac{EF}{BC}=\frac{DF}{AC}$.

To prove $\triangle DEF \sim \triangle ABC$.

ARGUMENT	REASONS
1. On DE lay off $DH=AB$, and on DF lay off $DK=AC$.	1. § 54, 14.
2. Draw HK.	2. § 54, 15.
3. $\frac{DE}{AB}=\frac{DF}{AC}$.	3. By hyp.
4. $\therefore \frac{DE}{DH}=\frac{DF}{DK}$.	4. § 309.
5. $\therefore HK \parallel EF$.	5. § 415.
6. $\therefore \triangle DEF \sim \triangle DHK$.	6. § 423.
It remains to prove $\triangle DHK = \triangle ABC$.	
7. $\frac{DE}{DH}=\frac{EF}{HK}$.	7. § 424, 2.
8. But $\frac{DE}{AB}=\frac{EF}{BC}$; *i.e.* $\frac{DE}{DH}=\frac{EF}{BC}$.	8. By hyp.
9. $\therefore HK=BC$.	9. § 391.
10 Now $DH=AB$ and $DK=AC$.	10. Arg. 1.
11. $\therefore \triangle DHK = \triangle ABC$.	11. § 116.
12. But $\triangle DEF \sim \triangle DHK$.	12. Arg. 6.
13. $\therefore \triangle DEF \sim \triangle ABC$. Q.E.D.	13. § 309.

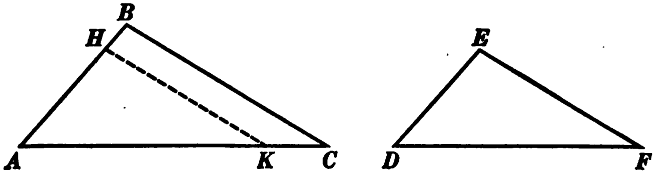
Ex. 673. If the sides of two triangles are 9, 12, 15, and 6, 8, 10 respectively, are the triangles similar? Explain.

Ex. 674. Construct a triangle that shall have a given perimeter and shall be similar to a given triangle.

Ex. 675. Construct a trapezoid, given the two bases and the two diagonals. Hint. How do the diagonals of a trapezoid divide each other?

Proposition XVI. Theorem

428. *If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, the triangles are similar.*



Given $\triangle$s ABC and DEF with $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

To prove $\triangle ABC \sim \triangle DEF$.

Argument	Reasons
1. Place $\triangle DEF$ on $\triangle ABC$ so that $\angle D$ shall coincide with $\angle A$, DE falling on AB, and DF on AC. Represent $\triangle DEF$ in its new position by $\triangle AHK$.	1. § 54, 14.
2 $\frac{AB}{DE} = \frac{AC}{DF}$.	2. By hyp.
3. $\therefore \frac{AB}{AH} = \frac{AC}{AK}$.	3. § 309.
4. $\therefore HK \parallel BC$.	4. § 415.
5. $\therefore \triangle ABC \sim \triangle AHK$.	5. § 423.
6. But $\triangle AHK$ is $\triangle DEF$ transferred to a different position.	6. Arg. 1.
7. $\therefore \triangle ABC \sim \triangle DEF$. Q.E.D.	7. § 309.

Ex. 676. Two triangles are similar if two sides and the median drawn to one of these sides in one triangle are proportional to two sides and the corresponding median in the other triangle.

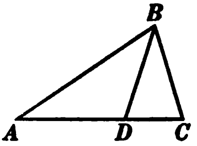


FIG. 1.

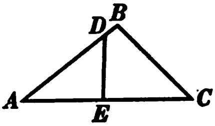


FIG. 2.

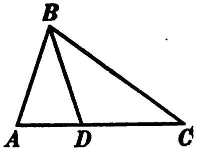


FIG. 3.

Ex. 677. In triangles ABC and DBC, Fig. 1, $AB = AC$ and $BD = BC$. Prove triangle ABC similar to triangle DBC.

Ex. 678. In Fig. 2, $AB : AC = AE : AD$. Prove triangle ABC similar to triangle ADE.

Ex. 679. If in triangle ABC, Fig. 3, $CA = BC$, and if D is a point such that $CA : AB = AB : AD$, prove $AB = BD$.

Ex. 680. Construct a triangle similar to a given triangle and having the sum of two sides equal to a given line.

PROPOSITION XVII. THEOREM

429. *Two triangles that have their sides parallel each to each, or perpendicular each to each, are similar.*

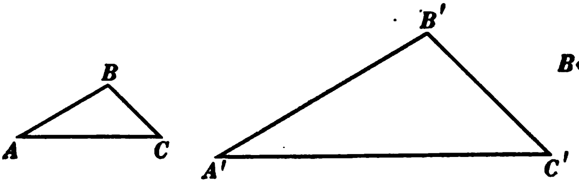


FIG. 1.

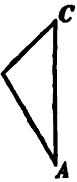


FIG. 2.

Given $\triangle\triangle\ ABC$ and $A'B'C'$, with AB, BC, and $CA \parallel$ (Fig. 1) or $\perp$ (Fig. 2) respectively to $A'B'$, $B'C'$, and $C'A'$.

To prove $\triangle ABC \sim \triangle A'B'C'$.

ARGUMENT	REASONS
1. AB, BC, and CA are $\parallel$ or $\perp$ respectively to $A'B'$, $B'C'$, and $C'A'$.	1. By hyp.
2. $\therefore \angle\angle\ A$, B, and C are equal respectively or are sup. respectively to $\angle\angle\ A'$, B', and C'.	2. §§ 198, 201

ARGUMENT	REASONS
3. Three suppositions may be made, therefore, as follows: (1) $\angle A + \angle A' = 2$ rt. $\angle$s, $\angle B + \angle B' = 2$ rt. $\angle$s, $\angle C + \angle C' = 2$ rt. $\angle$s. (2) $\angle A = \angle A'$, $\angle B + \angle B' = 2$ rt. $\angle$s, $\angle C + \angle C' = 2$ rt. $\angle$s. (3) $\angle A = \angle A'$, $\angle B = \angle B'$; hence, also, $\angle C = \angle C'$.	3. § 161, *a*.
4. According to (1) and (2) the sum of the $\angle$s of the two $\triangle$s is more than four rt. $\angle$s.	4. § 54, 2.
5. But this is impossible.	5. § 204.
6. $\therefore$ (3) is the only supposition admissible; *i.e.* the two $\triangle$s are mutually equiangular.	6. 161, *b*.
7. $\therefore \triangle ABC \sim \triangle A'B'C'$. Q.E.D.	7. § 420.

430. Question. Can one pair of angles in Prop. XVII be supplementary and the other two pairs equal?

SUMMARY OF CONDITIONS FOR SIMILARITY OF TRIANGLES

431. I. Two triangles are similar if they are mutually equiangular.

(*a*) Two triangles are similar if two angles of one are equal respectively to two angles of the other.

(*b*) Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

(*c*) If a line is drawn parallel to any side of a triangle, this line, with the other two sides, forms a triangle which is similar to the given triangle.

II. Two triangles are similar if their sides are proportional.

III. Two triangles are similar if they have an angle of one equal to an angle of the other, and the including sides proportional.

IV. Two triangles are similar if their sides are parallel each to each, or perpendicular each to each.

Ex. 681. Inscribe a triangle in a circle and circumscribe about the circle a triangle similar to the inscribed triangle.

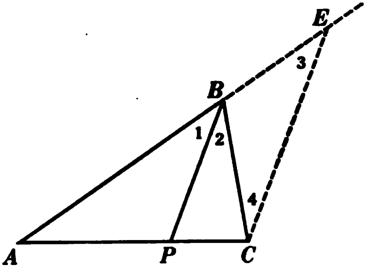
Ex. 682. Circumscribe a triangle about a circle and inscribe in the circle a similar triangle.

Ex. 683. The lines joining the mid-points of the sides of a triangle form a second triangle similar to the given triangle.

Ex. 684. ABC is a triangle inscribed in a circle. A line is drawn from A to P, any point of BC, and a chord is drawn from B to a point Q in arc BC so that angle ABQ equals angle APC. Prove $AB \times AC = AQ \times AP$.

Proposition XVIII. Theorem

432. ***The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.***



Given $\triangle ABC$ with BP the bisector of $\angle ABC$.

To prove $AP : PC = AB : BC$.

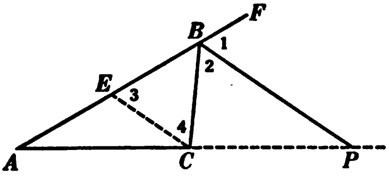
Argument	Reasons
1. Through C draw $CE \parallel PB$, meeting AB prolonged at E.	1. § 179.
2. In $\triangle AEC$, $AP : PC = AB : BE$.	2. § 410.
3. Now $\angle 3 = \angle 1$.	3. § 190.
4. And $\angle 4 = \angle 2$.	4. § 189.
5. But $\angle 1 = \angle 2$.	5. By hyp.
6. $\therefore \angle 3 = \angle 4$.	6. § 54, 1.
7. $\therefore BC = BE$.	7. § 162.
8. $\therefore AP : PC = AB : BC$. Q.E.D.	8. § 309.

Ex. 685. The sides of a triangle are 8, 12, and 15. Find the segments of side 8 made by the bisector of the opposite angle.

Ex. 686. In the triangle of Ex. 685, find the segments of sides 12 and 15 made by the bisectors of the angles opposite.

PROPOSITION XIX. THEOREM

433. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the other two sides.*



Given $\triangle ABC$, with BP the bisector of exterior $\angle CBF$.

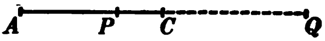
To prove $AP : PC = AB : BC$.

ARGUMENT	REASONS
1. Through C draw $CE \parallel PB$, meeting AB at E.	1. § 179.
2. Then in $\triangle ABP$, $AP : PC = AB : BE$.	2. § 409.
3. Now $\angle 3 = \angle 1$.	3. § 190.
4. And $\angle 4 = \angle 2$.	4. § 189.
5. But $\angle 1 = \angle 2$.	5. By hyp.
6. $\therefore \angle 3 = \angle 4$.	6. § 54, 1.
7. $\therefore BC = BE$.	7. § 162.
8. $\therefore AP : PC = AB : BC$.	8. § 309.
Q.E.D.	

Ex. 687. Compare the lettering of the figures for Props. XVIII and XIX, and also the steps in the argument. Could one argument serve for the two cases?

Ex. 688. The sides of a triangle are 9, 12, and 16. Find the segments of side 9 made by the bisector of the exterior angle at the opposite vertex.

434. Def. A line is **divided harmonically** if it is divided internally and externally into segments whose ratios are numerically equal; thus, if line AC is divided internally at P and externally at Q so that the ratio of AP to PC is numerically equal to the ratio of AQ to QC, AC is said to be divided harmonically.

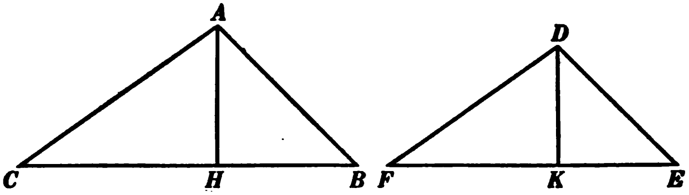


Ex. 689. The bisectors of the interior and exterior angles at any vertex of a triangle divide the opposite side harmonically.

Ex. 690. Divide a given straight line harmonically in the ratio of 3 to 5; in the ratio of a to b, where a and b are given straight lines.

PROPOSITION XX. THEOREM

435. *In two similar triangles any two homologous altitudes have the same ratio as any two homologous sides.*



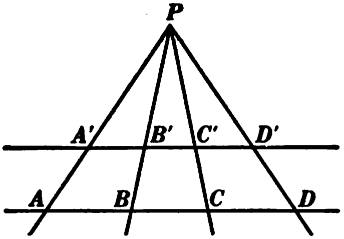
Given two similar $\triangle$ ABC and DEF, with two corresponding altitudes AH and DK.

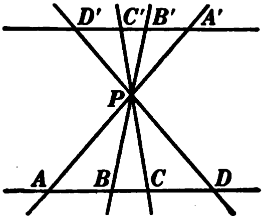
To prove $\frac{AH}{DK} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

ARGUMENT	REASONS
1. In rt. $\triangle$ ABH and DEK, $\angle B = \angle E$.	1. § 424, 1.
2. $\therefore \triangle ABH \sim \triangle DEK$.	2. § 422.
3. $\therefore \frac{AH\text{, opposite } \angle B}{DK\text{, opposite } \angle E} = \frac{AB\text{, opposite } \angle BHA}{DE\text{, opposite } \angle EKD}$.	3. § 424, 2.
4. But $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.	4. § 424, 2.
5. $\therefore \frac{AH}{DK} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. Q.E.D	5. § 54, 1.

PROPOSITION XXI. THEOREM

436. *If three or more straight lines drawn through a common point intersect two parallels, the corresponding segments of the parallels are proportional.*





Given lines PA, PB, PC, PD drawn through a common point P and intersecting the ∥ lines AD and $A'D'$ at points A, B, C, D and A', B', C', D', respectively.

To prove $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$.

ARGUMENT	REASONS
1. $AD \parallel A'D'$.	1. By hyp.
2. $\therefore \triangle APB \sim \triangle A'PB'$.	2. § 423.
3. $\therefore \frac{AB}{A'B'} = \frac{PB}{PB'}$.	3. § 424, 2.
4. Likewise $\triangle BPC \sim \triangle B'PC'$.	4. Args. 1–2.
5. And $\frac{BC}{B'C'} = \frac{PB}{PB'}$.	5. § 424, 2.
6. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'}$.	6. § 54, 1.
7. Likewise it can be proved that $\frac{BC}{B'C'} = \frac{CD}{C'D'}$.	7. By steps similar to 1–6.
8. $\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. Q.E.D.	8. § 54, 1.

Ex. 691. If three or more non-parallel straight lines intercept proportional segments on two parallels, they pass through a common point.

Ex. 692. A man is riding in an automobile at the uniform rate of 30 miles an hour on one side of a road, while on a footpath on the other side a man is walking in the opposite direction. If the distance between the footpath and the auto track is 44 feet, and a tree 4 feet from the footpath continually hides the chauffeur from the pedestrian, does the pedestrian walk at a uniform rate? If so, at what rate does he walk?

Ex. 693. Two sides of a triangle are 8 and 11, and the altitude upon the third side is 6. A similar triangle has the side homologous to 8 equal to 12. Compute as many parts of the second triangle as you can.

Ex. 694. In two similar triangles, any two homologous bisectors are in the same ratio as any two homologous sides.

Ex. 695. In two similar triangles, any two homologous medians are in the same ratio as any two homologous sides.

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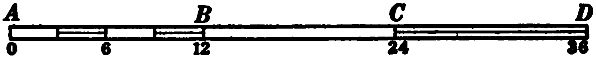
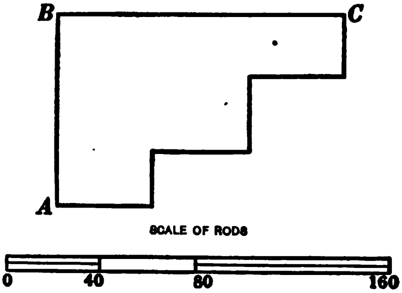
437. Drawing to Scale. Measure the top of your desk. Make a drawing on paper in which each line is $\frac{1}{12}$ as long as the corresponding line of your desk. Check your work by measuring the diagonal of your drawing, and the corresponding line of your desk. This is called **drawing to scale.** Map drawing is a common illustration of this principle. The *scale* of the drawing may be represented: (1) by saying, "Scale, $\frac{1}{12}$" or "Scale, 1 inch to 12 inches"; (2) by actually drawing the scale as indicated.



Ex. 696. Using the scale above, draw lines on paper to represent 24 inches; 3 feet 3 inches.

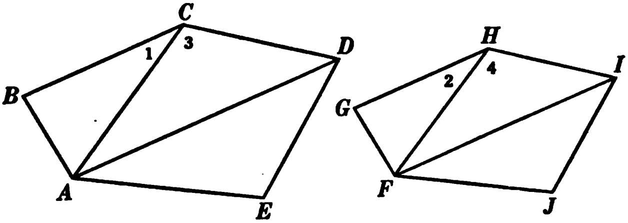
Ex. 697. On the blackboard draw, to the scale above, a circle whose diameter is 28 feet.

Ex. 698. The figure represents a farm drawn to the scale indicated. Find the cost of putting a fence around the farm, if the fencing costs $.25 per rod.



PROPOSITION XXII. THEOREM

438. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



Given polygons $ABCDE$ and $FGHIJ$ with $\triangle ABC \sim \triangle FGH$, $\triangle ACD \sim \triangle FHI$, $\triangle ADE \sim \triangle FIJ$.

To prove polygon $ABCDE \sim$ polygon $FGHIJ$.

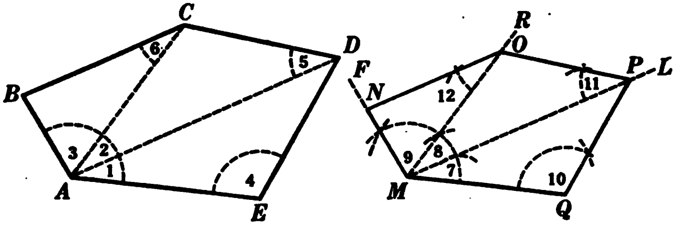
ARGUMENT	REASONS
1. In $\triangle$ ABC and FGH, $\angle B = \angle G$.	1. § 424, 1.
2. Also $\angle 1 = \angle 2$.	2. § 424, 1.
3. In $\triangle$ ACD and FHI, $\angle 3 = \angle 4$.	3. § 424, 1.
4. $\therefore \angle 1 + \angle 3 = \angle 2 + \angle 4$.	4. § 54, 2.
5. $\therefore \angle BCD = \angle GHI$.	5. § 309.
6. Likewise $\angle CDE = \angle HIJ$, $\angle E = \angle J$, and $\angle EAB = \angle JFG$.	6. By steps similar to 1–5.
7. $\therefore$ polygons $ABCDE$ and $FGHIJ$ are mutually equiangular.	7. By proof.
8. In $\triangle$ ABC and FGH, $\frac{AB}{FG} = \frac{BC}{GH} = \frac{CA}{HF}$.	8. § 424, 2.
9. In $\triangle$ ACD and FHI, $\frac{CA}{HF} = \frac{CD}{HI} = \frac{DA}{IF}$.	9. § 424, 2.
10. And in $\triangle$ ADE and FIJ, $\frac{DA}{IF} = \frac{DE}{IJ} = \frac{EA}{JF}$.	10. § 424, 2.
11. $\therefore \frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HI} = \frac{DE}{IJ} = \frac{EA}{JF}$.	11. § 54, 1.
12. $\therefore$ polygon $ABCDE \sim$ polygon $FGHIJ$.	12. § 419.

Q.E.D.

439. Cor. *Any two similar polygons may be divided into the same number of triangles similar each to each and similarly placed.*

PROPOSITION XXIII. PROBLEM

440. *Upon a line homologous to a side of a given polygon, to construct a polygon similar to the given polygon.*



Given polygon AD and line MQ homol. to side AE.

To construct, on MQ, a polygon $\sim$ polygon AD.

I. Construction

1. Draw all possible diagonals from A, as AC and AD.

2. At M, beginning with MQ as a side, construct $\angle$s 7, 8, and 9 equal respectively to $\angle$s 1, 2, and 3. § 125.

3. At Q, with MQ as a side, construct $\angle 10$ equal to $\angle 4$, and prolong side QP until it meets ML at P. § 125.

4. At P, with PM as a side, construct $\angle 11$ equal to $\angle 5$, and prolong side PO until it meets MR at O. § 125.

5. At O, with OM as a side, construct $\angle 12$ equal to $\angle 6$, and prolong side ON until it meets MF at N. § 125.

6. MP is the polygon required.

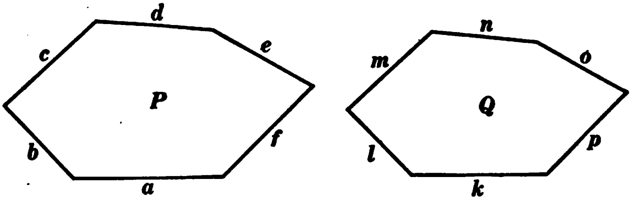
II. Proof

ARGUMENT	REASONS
1. $\triangle ADE \sim \triangle MPQ$, $\triangle ACD \sim \triangle MOP$, and $\triangle ABC \sim \triangle MNO$.	1. § 421.
2. $\therefore$ polygon $MP \sim$ polygon AD. Q.E.D	2. § 438.

III. The discussion is left as an exercise for the student.

PROPOSITION XXIV. THEOREM

441. *The perimeters of two similar polygons are to each other as any two homologous sides.*



Given $\sim$ polygons P and Q, with sides a, b, c, d, e, and f homol. respectively to sides k, l, m, n, o, and p.

To prove $\dfrac{\text{perimeter of } P}{\text{perimeter of } Q} = \dfrac{a}{k}$.

ARGUMENT	REASONS
1. $\dfrac{a}{k} = \dfrac{b}{l} = \dfrac{c}{m} = \dfrac{d}{n} = \dfrac{e}{o} = \dfrac{f}{p}$	1. § 419.
2. $\therefore \dfrac{a+b+c+d+e+f}{k+l+m+n+o+p} = \dfrac{a}{k}$	2. § 401.
3. That is, $\dfrac{\text{perimeter of } P}{\text{perimeter of } Q} = \dfrac{a}{k}$. Q.E.D.	3. § 309.

Ex. 699. The perimeters of two similar polygons are 152 and 138 a side of the first is 8. Find the homologous side of the second.

Ex. 700. The perimeters of two similar polygons are to each other as any two homologous diagonals.

Ex. 701. The perimeters of two similar triangles are to each other as any two homologous medians.

Ex. 702. If perpendiculars are drawn to the hypotenuse of a right triangle at its extremities, and if the other two sides of the triangle are prolonged to meet these perpendiculars, the figure thus formed contains five triangles each of which is similar to any one of the others.

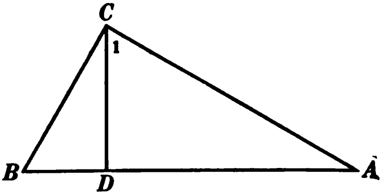
PROPOSITION XXV. THEOREM

442. *In a right triangle, if the altitude upon the hypotenuse is drawn:*

I. *The triangles thus formed are similar to the given triangle and to each other.*

II. *The altitude is a mean proportional between the segments of the hypotenuse.*

III. *Either side is a mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to that side.*



Given rt. $\triangle ABC$ and the altitude CD upon the hypotenuse.

To prove:

I. $\triangle BCD \sim \triangle ABC$, $\triangle ADC \sim \triangle ABC$, and $\triangle BCD \sim \triangle ADC$

II. $BD : CD = CD : DA$.

III. $AB : BC = BC : DB$ and $AB : AC = AC : AD$.

I.	ARGUMENT	REASONS
1	In rt. $\triangle$s BCD and ABC, $\angle B = \angle B$.	1. By iden.
2.	$\therefore \triangle BCD \sim \triangle ABC$.	2. § 422.
3.	In rt. $\triangle$s ADC and ABC, $\angle A = \angle A$.	3. By iden.
4.	$\therefore \triangle ADC \sim \triangle ABC$.	4. § 422.
5.	$\therefore \angle 1 = \angle B$.	5. § 424, 1.
6.	$\therefore \triangle BCD \sim \triangle ADC$. Q.E.D.	6. § 422.

II. The proof of II is left as an exercise for the student.

HINT. Mark (with colored chalk, if convenient) the lines required in the proportion. Decide which triangles will furnish these lines, and use the fact that homologous sides of similar triangles are proportional.

III. The proof of III is left as an exercise for the student.

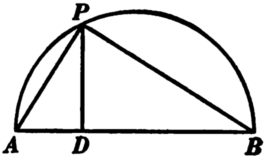
Hint. Use the same method as for II.

443. Cor. I. *In a right triangle, if the altitude upon the hypotenuse is drawn:*

I. *The square of the altitude is equal to the product of the segments of the hypotenuse.* Thus, $\overline{CD}^2 = BD \cdot DA$. (See § 442, II.)

II. *The square of either side is equal to the product of the whole hypotenuse and the segment of the hypotenuse adjacent to that side.* Thus, $\overline{BC}^2 = AB \cdot DB$, and $\overline{AC}^2 = AB \cdot AD$. (See § 442, III.)

444. Cor. II. *If from any point in the circumference of a circle a perpendicular to a diameter is drawn, and if chords are drawn from the point to the ends of the diameter:*



I. *The perpendicular is a mean proportional between the segments of the diameter.*

II. *Either chord is a mean proportional between the whole diameter and the segment of the diameter adjacent to the chord.*

Hint. △ APB is a rt. △. Apply Prop. XXV, II and III.

Ex. 703. In a right triangle, the squares of the two sides are proportional to the segments of the hypotenuse made by the altitude upon it.

Hint. Apply § 443, II.

Ex. 704. The sides of a right triangle are 9, 12, and 15.

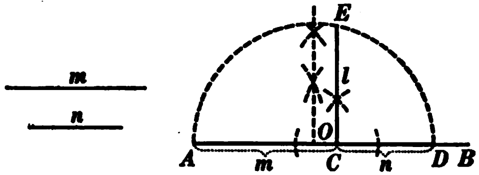
(1) Compute the segments of the hypotenuse made by the altitude upon it.

(2) Compute the length of the altitude.

Ex. 705. If the two arms of a right triangle are 6 and 8, compute the length of the perpendicular from the vertex of the right angle to the hypotenuse.

Proposition XXVI. Problem

445. *To construct a mean proportional between two given lines.*



Given two lines, m and n.

To construct a line l so that $m : l = l : n$.

I. Construction

1. On any indefinite str. line, as AB, lay off $AC = m$ and $CD = n$.

2. With O, the mid-point of AD, as center and with a radius equal to OD, describe a semicircumference.

3. At C construct $CE \perp AD$, meeting the semicircumference at E. § 148.

4. CE is the required line l.

II. The proof and discussion are left as an exercise for the student.

Ex. 706. By means of § 444, II, construct a mean proportional between two given lines by a method different from that given in Prop. XXVI.

Ex. 707. Use the method of Prop. XXVI to construct a line equal to $\sqrt{3\,ab}$, a and b being given lines.

Analysis. Let $x =$ the required line.

Then $x = \sqrt{3\,ab}$.

$$\therefore x^2 = 3\,ab.$$

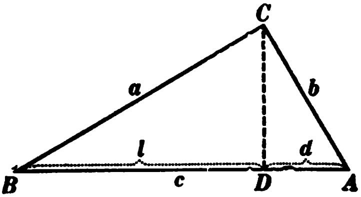
$$\therefore 3\,a : x = x : b.$$

Ex. 708. Construct a line equal to $a\sqrt{3}$, where a is a given line.

Ex. 709. Using a line one inch long as a unit, construct a line equal to $\sqrt{3}$; $\sqrt{5}$; $\sqrt{6}$. Choosing your own unit, construct a line equal to $3\sqrt{2}$, $2\sqrt{3}$, $5\sqrt{5}$.

PROPOSITION XXVII. THEOREM

446. *In any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.*



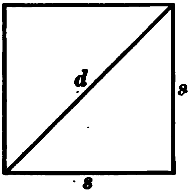
Given rt. △ ABC, with its rt. ∠ at C.

To prove $a^2 + b^2 = c^2$.

ARGUMENT	REASONS
1. From C draw $CD \perp AB$ forming segments l and d.	1. § 155.
2. $a^2 = c \cdot l$.	2. § 413, II.
3. $b^2 = c \cdot d$.	3. § 443, II.
4. $\therefore a^2 + b^2 = c(l + d)$.	4. § 54, 2.
5. $\therefore a^2 + b^2 = c \cdot c = c^2$. Q.E.D.	5. § 309.

447. Cor. I. *The square of either side of a right triangle is equal to the square of the hypotenuse minus the square of the other side.*

448. Cor. II. *The diagonal of a square is equal to its side multiplied by the square root of two.*



OUTLINE OF PROOF

1. $d^2 = s^2 + s^2 = 2\,s^2$.
2. $\therefore d = s\sqrt{2}$. Q.E.D.

449. Historical Note. The property of the right triangle stated in Prop. XXVII was known at a very early date, the ancient Egyptians, 2000 B.C., having made a right triangle by stretching around three pegs a cord measured off into 3, 4, and 5 units. See Note, Book IV, § 510.

Ex. 710. By means of § 448, construct a line equal to $\sqrt{2}$ inches.

Ex. 711. If a side of a square is 6 inches, find its diagonal.

Ex. 712. The hypotenuse of a right triangle is 15 and one arm is 9. Find the other arm and the segments of the hypotenuse made by the perpendicular from the vertex of the right angle.

Ex. 713. Find the altitude of an equilateral triangle whose side is 6 inches.

Ex. 714. Find a side of an equilateral triangle whose altitude is 8 inches.

Ex. 715. Divide a line into segments which shall be in the ratio of 1 to $\sqrt{2}$.

Ex. 716. The radius of a circle is 10 inches. Find the length of a chord 6 inches from the center; 4 inches from the center.

Ex. 717. The radius of a circle is 20 inches. How far from the center is a chord whose length is 32 inches? whose length is 28 inches?

Ex. 718. In a circle a chord 24 inches long is 5 inches from the center. How far from the center is a chord whose length is 12 inches?

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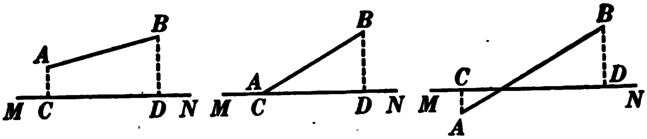
450. Def. The **projection of a point** upon a line is the foot of the perpendicular from the point to the line.

451. Def. The **projection of a line segment** upon a line is the segment of the second line included between the projections of the extremities of the first line upon the second.

Thus, *C* is the projection of *A* upon *MN*, *D* is the projection of *B* upon *MN*, and *CD* is the projection of *AB* upon *MN*.



Ex. 719. In the figures above, under what condition will the projection of *AB* on *MN* be a maximum? a minimum? Will the projection *CD* ever be equal to *AB*? greater than *AB*? Will the projection ever be a point?

Ex. 720. In a right isosceles triangle the hypotenuse of which is 10 inches, find the length of the projection of either arm upon the hypotenuse.

Ex. 721. Find the projection of one side of an equilateral triangle upon another if each side is 6 inches.

Ex. 722. Draw the projections of the shortest side of a triangle upon each of the other sides: (1) in an acute triangle; (2) in a right triangle; (3) in an obtuse triangle. Draw the projections of the longest side in each case.

Ex. 723. Two sides of a triangle are 8 and 12 inches and their included angle is 60°. Find the projection of the shorter upon the longer.

Ex. 724. In Ex. 723, find the projection of the shorter side upon the longer if the included angle is 30°; 45°.

Ex. 725. Parallel lines that have equal projections on the same line are equal.

PROPOSITION XXVIII. PROBLEM

452. *In any triangle to find the value of the square of the side opposite an acute angle in terms of the other two sides and of the projection of either of these sides upon the other.*

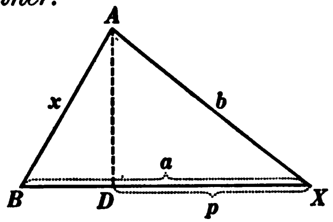


FIG. 1.

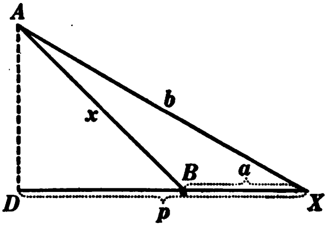


FIG. 2.

Given $\triangle BAX$, with $\angle X$ acute; and p, the projection of b upon a.

To find the value of x^2 in terms of a, b, and p.

ARGUMENT	REASONS
1. In rt. $\triangle BAD$, $x^2 = \overline{AD}^2 + \overline{DB}^2$.	1. § 446.
2. But $\overline{AD}^2 = b^2 - p^2$.	2. § 447.
3. And $DB = a - p$ (Fig. 1) or $p - a$ (Fig. 2).	3. § 54, 11.
4. $\therefore \overline{DB}^2 = a^2 - 2\,ap + p^2$.	4. § 54, 13.
5. $\therefore x^2 = b^2 - p^2 + a^2 - 2\,ap + p$; i.e. $x^2 = a^2 + b^2 - 2\,ap$. Q.E.F.	5. § 309.

453. Question. Why is it not necessary to include here the figure and discussion for a right triangle?

454. Prop. XXVIII may be stated in the form of a theorem as follows:

In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of these sides and the projection of the other side upon it.

Ex. 726. If the sides of a triangle are 7, 8, and 10, is the angle opposite 10 obtuse, right, or acute? Why?

Ex. 727. Apply the statement of Prop. XXVIII to the square of an arm of a right triangle.

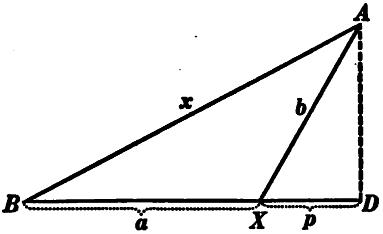
Ex. 728. Find x (in the figure for Prop. XXVIII) in terms of a and b and the projection of a upon b.

Ex. 729. If the sides of a triangle are 13, 14, and 15, find the projection of the first side upon the second.

Ex. 730. If two sides of a triangle are 4 and 12 and the projection of the first side upon the second is 2, find the third side of the triangle.

PROPOSITION XXIX. THEOREM

455. *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of these sides and the projection of the other side upon it.*



Given $\triangle BAX$ with $\angle X$ obtuse, and p, the projection of b upon a.

To prove $x^2 = a^2 + b^2 + 2ap$.

ARGUMENT	REASONS
1. In rt. △ BAD, $x^2 = \overline{AD}^2 + \overline{DB}^2$.	1. § 446.
2. But $\overline{AD}^2 = b^2 - p^2$	2. § 447.
3. And $DB = a + p$.	3. § 54, 11.
4. $\therefore \overline{DB}^2 = a^2 + 2ap + p^2$.	4. § 54, 13.
5. $\therefore x^2 = b^2 - p^2 + a^2 + 2ap + p^2$; i.e. $x^2 = a^2 + b^2 + 2ap$. Q.E.D.	5. § 309.

456. From Props. XXVIII and XXIX, we may derive the following formulas for computing the projection of one side of a triangle upon another; thus if *a*, *b*, and *c* represent the sides of a triangle:

From Prop. XXVIII, $$p = \frac{a^2 + b^2 - c^2}{2a}. \qquad (1)$$

From Prop. XXIX, $$-p = \frac{a^2 + b^2 - c^2}{2a}. \qquad (2)$$

It is seen that the second members of these two equations are identical and that the first members differ only in sign. Hence, formula (1) may always be used for computing the length of a projection. It need only be remembered that if p is positive in any calculation, it indicates that the angle opposite c is acute; while if p is negative, the angle opposite c is obtuse. It can likewise be shown (see Prop. XXVII) that if $p = 0$, the angle opposite c is a right angle.

Ex. 731. Write the formula for the projection of a upon b.

Ex. 732. In triangle ABC, $a = 15$, $b = 20$, $c = 25$; find the projection of b upon c. Is angle A acute, right, or obtuse?

Ex. 733. In the triangle of Ex. 732, find the projection of a upon b. Is angle C acute, right, or obtuse?

Ex. 734. The sides of a triangle are 8, 14, and 20. Is the angle opposite the side 20 acute, right, or obtuse?

Ex. 735 If two sides of a triangle are 10 and 12, and their included angle is 120°, what is the value of the third side?

Ex. 736. If two sides of a triangle are 12 and 16, and their included angle is 45°, find the third side

Ex. 737. If in triangle ABC, angle $C = 120°$, prove that

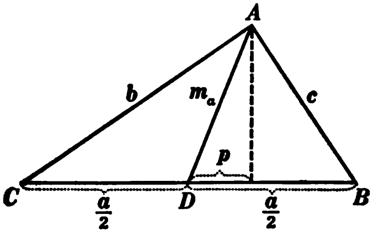
$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \cdot BC.$$

Ex. 738. If a line is drawn from the vertex C of an isosceles triangle ABC, meeting base AB prolonged at D, prove that

$$\overline{CD}^2 - \overline{CB}^2 = AD \cdot BD.$$

PROPOSITION XXX. THEOREM

457. *In any triangle, the sum of the squares of any two sides is equal to twice the square of half the third side increased by twice the square of the median upon that side.*



Given $\triangle ABC$ with m_a, the median to side a.

To prove $b^2 + c^2 = 2\left(\frac{a}{2}\right)^2 + 2\,m_a^2.$

ARGUMENT	REASONS
1. Suppose $b > c$; then $\angle ADC$ is obtuse and $\angle BDA$ is acute.	1. § 173.
2. Let p be the projection of m_a upon a.	2. § 451.
3. Then from $\triangle ADC$, $b^2 = \left(\frac{a}{2}\right)^2 + m_a^2 + 2\left(\frac{a}{2}\right)p.$	3. § 455.
4. And from $\triangle ABD$, $c^2 = \left(\frac{a}{2}\right)^2 + m_a^2 - 2\left(\frac{a}{2}\right)p.$	4. § 454.
5. $\therefore b^2 + c^2 = 2\left(\frac{a}{2}\right)^2 + 2\,m_a^2.$ Q.E.D.	5. § 54, 2.

458. Cor. *The difference of the squares of two sides of any triangle is equal to twice the product of the third side and the projection of the median upon the third side.*

HINT. Subtract the equation in Arg. 4 from that in Arg. 3 (§ 457) member from member.

Ex. 739. Write the formula involving the median to b; to c.

Ex. 740. Apply Prop. XXX to a triangle right-angled at A; at B; at C.

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459. It will be seen that the formula of Prop. XXX contains the three sides of a triangle and a median to one of these sides. Hence, if the three sides of a triangle are given, the median to any one of them can be found; also, if two sides and any median are given, the third side can be found.

Ex. 741. If the sides of a triangle ABC are 5, 7, and 8, find the lengths of the three medians.

Ex. 742. If the sides of a triangle are 12, 16, and 20, find the median to side 20. How does it compare in length with the side to which it is drawn? Why?

Ex. 743. In triangle ABC, $a = 16$, $b = 22$, and $m_c = 17$. Find c.

Ex. 744. In a right triangle, right-angled at C, $m_c = 8\frac{1}{2}$; what is c? Find one pair of values for a and b that will satisfy the conditions of the problem.

Ex. 745. The sum of the squares of the four sides of any parallelogram is equal to the sum of the squares of its diagonals.

Ex. 746. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of its diagonals increased by four times the square of the line joining their mid-points.

Ex. 747. Construct a triangle ABC, given b, c, and m_a.

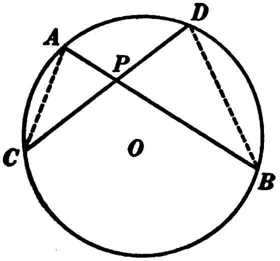
Ex. 748. Construct a triangle ABC, given a, b, and the projection of b upon a.

Ex. 749. Compute the side of a rhombus whose diagonals are 12 and 16.

Ex. 750. If the square of the longest side of a triangle is greater than the sum of the squares of the other two sides, is the triangle obtuse. right, or acute? Why?

PROPOSITION XXXI. THEOREM

460. *If through a point within a circle two chords are drawn, the product of the two segments of one of these chords is equal to the product of the two segments of the other.*



Given P, a point within circle O, and AB and CD, any two chords drawn through P.

To prove $PA \cdot PB = PC \cdot PD$.

The proof is left as an exercise for the student.

HINT. Prove $\triangle APC \sim \triangle PDB$.

Ex. 751. In the figure for Prop. XXXI, if $PA = 5$, $PB = 12$, and $PD = 6$, find PC.

Ex. 752. In the same figure, if $PC = 10$, $PD = 8$, and $AB = 21$, find PA and PB.

Ex. 753. In the same figure, if $PC = 6$, $DC = 22$, and $AB = 20$, find AP and PB.

Ex. 754. In the same figure, if $PA = m$, $PC = n$, and $PD = r$, find PB.

Ex. 755. If two chords intersecting within a circle are of lengths 8 and 10, and the second bisects the first, what are the segments of the second?

Ex. 756. By means of Prop. XXXI construct a mean proportional between two given lines.

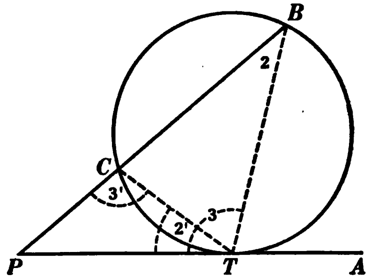
Ex. 757. If two chords intersect within a circle and the segments of one chord are a and b inches, while the second chord measures d inches, construct the segments of the second chord.

HINT. Find the locus of the mid-points of chords equal to d.

Ex. 758. If two lines AB and CD intersect at E so that $AE \cdot EB = CE \cdot ED$, then a circumference can be passed through the four points A, B, C, D.

PROPOSITION XXXII. THEOREM

461. *If a tangent and a secant are drawn from any given point to a circle, the tangent is a mean proportional between the whole secant and its external segment.*



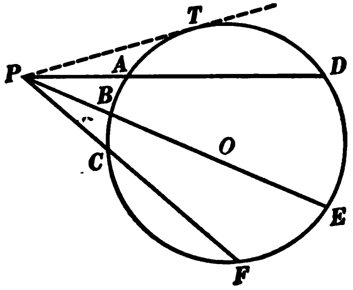
Given tangent PT and secant PB drawn from the point P to the circle O.

To prove $\frac{PB}{PT} = \frac{PT}{PC}$.

	ARGUMENT	REASONS
1	Draw CT and BT.	1. § 54, 15.
2.	In $\triangle$s PBT and CTP, $\angle P = \angle P$.	2. By iden.
3.	$\angle 2 = \angle 2'$.	3. § 362, *a*.
4.	$\therefore \triangle PBT \sim \triangle CTP$.	4. § 421.
5.	$\therefore \frac{PB\text{, opposite } \angle 3 \text{ in } \triangle PBT}{PT\text{, opposite } \angle 3' \text{ in } \triangle CTP} = \frac{PT\text{, opposite } \angle 2 \text{ in } \triangle PBT}{PC\text{, opposite } \angle 2' \text{ in } \triangle CTP}$. Q.E.D.	5. § 424, 2.

462. Cor. I. *If a tangent and a secant are drawn from any given point to a circle, the square of the tangent is equal to the product of the whole secant and its external segment.*

463. Cor. II. *If two or more secants are drawn from any given point to a circle, the product of any secant and its external segment is constant.*



Given secants PD, PE, PF, drawn from point P to circle O, and let their external segments be denoted by PA, PB, PC, respectively.

To prove $PD \cdot PA = PE \cdot PB = PF \cdot PC$.

ARGUMENT	REASONS
1. From P draw a tangent to circle O, as PT.	1. § 286.
2. $\overline{PT}^2 = PD \cdot PA$; $\overline{PT}^2 = PE \cdot PB$; $\overline{PT}^2 = PF \cdot PC$.	2. § 462.
3. $\therefore PD \cdot PA = PE \cdot PB = PF \cdot PC$. Q.E.D.	3. § 54, 1.

Ex. 759. If a tangent and a secant drawn from the same point to a circle measure 6 and 18 inches, respectively, how long is the external segment of the secant?

Ex. 760. Two secants are drawn to a circle from an outside point. If their external segments are 12 and 9, and the internal segment of the first secant is 8, what is the length of the second secant?

Ex. 761. The tangents to two intersecting circles from any point in their common chord (prolonged) are equal.

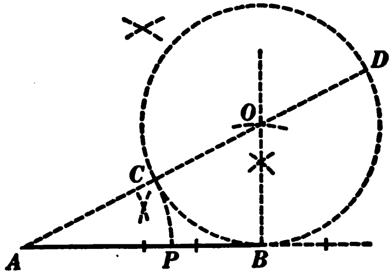
Ex. 762. If two circumferences intersect, their common chord (prolonged) bisects their common tangents.

464. Def. A line is said to be divided in **extreme and mean ratio** if it is divided into two parts such that one part is a mean proportional between the whole line and the other part.

Thus, AB is divided in extreme and mean ratio at P if $AB : AP = AP : PB$. This division is known as the **golden section**.

PROPOSITION XXXIII. PROBLEM

465. *To divide a line internally in extreme and mean ratio.*



Given line AB.

To find, in AB, a point P such that $AB : AP = AP : PB$.

I. Construction

1. From B draw $BO \perp AB$ and $= \frac{1}{2} AB$. § 148.

2. With O as center and with BO as radius describe a circumference.

3. From A draw a secant through O, cutting the circumference at C and D.

4. With A as center and with AC as radius draw $\overset{\frown}{CP}$, cutting AB at P.

5. $AB : AP = AP : PB$.

II. Proof

ARGUMENT	REASONS
1. AB is tangent to circle O.	1. § 314.
2. $\therefore AD : AB = AB : AC$.	2. § 461.
3. $\therefore AD - AB : AB = AB - AC : AC$.	3. § 399.
4. $\therefore AD - CD : AB = AB - AP : AP$.	4. § 309.
5. $\therefore AP : AB = PB : AP$.	5. § 309.
6. $\therefore AB : AP = AP : PB$. Q.E.D.	6. § 395.

III. The discussion is left as an exercise for the student.

Ex. 763. Divide a line AB externally in extreme and mean ratio.

HINT. In the figure for Prop. XXXIII prolong BA to P', making $P'A = AD$. Then prove $AB : P'A = P'A : P'B$.

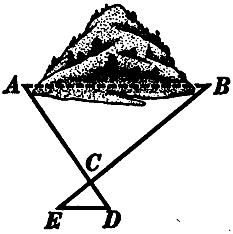
Ex. 764. If the line l is divided internally in extreme and mean ratio, and if s is the greater segment, find the value of s in terms of l.

HINT. $l : s = s : l - s$.

Ex. 765. A line 10 inches long is divided internally in extreme and mean ratio. Find the lengths of the two segments.

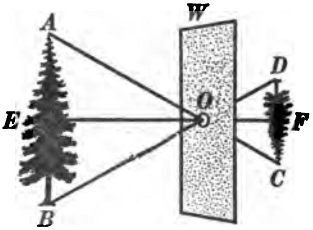
Ex. 766. A line 8 inches long is divided externally in extreme and mean ratio. Find the length of the longer segment.

MISCELLANEOUS EXERCISES



Ex. 767. Explain how the accompanying figure can be used to find the distance from A to B on opposite sides of a hill. $CE = \frac{1}{4}BC$, $CD = \frac{1}{4}AC$. ED is found by measurement to be 125 feet. What is the distance AB?

Ex. 768. A little boy wished to obtain the height of a tree in his yard. He set up a vertical pole 6 feet high and watched until the shadow of the pole measured exactly 6 feet. He then measured quickly the length of the tree's shadow and called this the height of the tree. Was his answer correct? Draw figures and explain. Use this method for measuring the height of your school building and flag pole.



Ex. 769. If light from a tree, as AB, is allowed to pass through a small aperture O, in a window shutter W, and strike a white screen or wall, an inverted image of the tree, as CD, is formed on the screen. If the distance $OE = 30$ feet, $OF = 8$ feet, and the length of the tree $AB = 35$ feet, find the length of the image CD. Under what condition will the length of the image equal the length of the tree?

This exercise illustrates the principle of the photographer's camera.

Ex. 770. By means of Prop. XXXII construct a mean proportional between two given lines.

Ex. 771. In a certain circle a chord $5\sqrt{5}$ inches from the center is 20 inches in length. Find the length of a chord 9 inches from the center.

Ex. 772. Compute the length of: (1) the common external tangent, (2) the common internal tangent, to two circles whose radii are 8 and 6, respectively, and the distance between whose centers is 20.

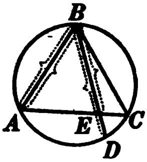
Ex. 773. If the hypotenuse of an isosceles right triangle is 16 inches, what is the length of each arm?

Ex. 774. If from a point a tangent and a secant are drawn and the segments of the secant are 4 and 12, how long is the tangent?

Ex. 775. Given the equation $\frac{m+n}{x+c}=\frac{2m}{c}$; solve for x.

Ex. 776. Find a mean proportional between $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

Ex. 777. The mean proportional between two unequal lines is less than half their sum.



Ex. 778. The diagonals of a trapezoid divide each other into segments which are proportional.

Ex. 779. ABC is an isosceles triangle inscribed in a circle. Chord BD is drawn from the vertex B, cutting the base in any point, as E. Prove $BD:AB=AB:BE$.

Ex. 780. In a triangle ABC the side AB is 305 feet. If a line parallel to BC divides AC in the ratio of 2 to 3, what are the lengths of the segments into which it divides AB?

Ex. 781. Construct, in one figure, four lines whose lengths shall be that of a given unit multiplied by $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$, respectively.

Ex. 782. Two sides of a triangle are 12 and 18 inches, and the perpendicular upon the first from the opposite vertex is 9 inches. What is the length of the altitude upon the second side?

Ex. 783. If $a:b=c:d$, show that

$$\left(a+\frac{l}{m}a\right):\left(b+\frac{l}{m}b\right)=\left(c+\frac{p}{q}c\right):\left(d+\frac{p}{q}d\right).$$

Also translate this fact into a verbal statement.

Ex. 784. If a constant is added to or subtracted from each term of a proportion, will the resulting numbers be in proportion? Give proof.

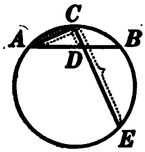
Ex. 785. If $r:s=t:q$, is $3r+\frac{r}{2}:s=7t:2q$? Prove your answer.

Ex. 786. One segment of a chord drawn through a point 7 units from the center of a circle is 4 units long. If the diameter of the circle is 15 units, what is the length of the other segment?

Ex. 787. The non-parallel sides of a trapezoid and the line joining the mid-points of the parallel sides, if prolonged, are concurrent.

Ex. 788. Construct a circle which shall pass through two given points and be tangent to a given straight line.

Ex. 789. The sides of a triangle are 10, 12, 15. Compute the lengths of the two segments into which the least side is divided by the bisector of the opposite angle.



Ex. 790. AB is a chord of a circle, and CE is any chord drawn through the middle point C of arc AB, cutting chord AB at D. Prove $CE : CA = CA : CD$.

Ex. 791. Construct a right triangle, given its perimeter and an acute angle.

Ex. 792. The base of an isosceles triangle is a, and the perpendicular let fall from an extremity of the base to the opposite side is b. Find the lengths of the equal sides.

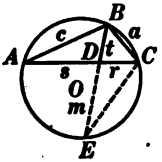
Ex. 793. AD and BE are two altitudes of triangle CAB. Prove that $AD : BE = CA : BC$.

Ex. 794. If two circles touch each other, their common external tangent is a mean proportional between their diameters.

Ex. 795. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.

Ex. 796. If three circles intersect each other, their common chords pass through a common point.

Ex. 797. The square of the bisector of an angle of a triangle is equal to the product of the sides of this angle diminished by the product of the segments of the third side made by the bisector.

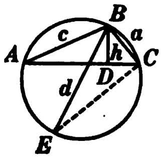


Given $\triangle ABC$ with t, the bisector of $\angle B$, dividing side b into the two segments s and r.

To prove $t^2 = ac - rs$.

Outline of Proof

1. Prove that $a : t = t + m : c$.
2. Then $ac = t^2 + tm = t^2 + rs$.
3. $\therefore t^2 = ac - rs$.



Ex. 798. In any triangle the product of two sides is equal to the product of the altitude upon the third side and the diameter of the circumscribed circle.

Hint. Prove $\triangle ABD \sim \triangle EBC$. Then prove

$$ac = hd.$$

BOOK IV

AREAS OF POLYGONS

466. A surface may be **measured** by finding how many times it contains a *unit of surface.* The unit of surface most frequently chosen is a square whose side is of unit length. If the unit length is an inch, the unit of surface is a square whose side is an inch. Such a unit is called a **square inch.** If the unit length is a foot, the unit of surface is a square whose side is a foot, and the unit is called a square foot.

467. Def. The result of the measurement is a **number,** which is called the **measure-number,** or **numerical measure,** or **area** of the surface.

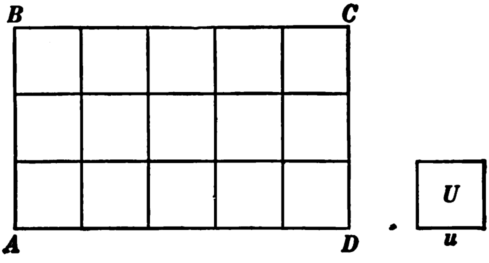


Fig. 1. Rectangle $AC = 15\,U$.

468. Thus, if the square U is contained in the rectangle $ABCD$ (Fig. 1) 15 times, then the measure-number or area of rectangle $ABCD$, in terms of U, is 15. If the given square is not contained in the given rectangle an integral number of

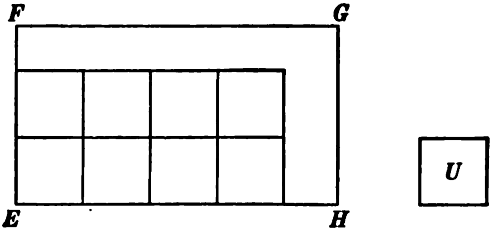


FIG. 2. Rectangle $EG = 8\ U+$.

times without a remainder (see Fig. 2), then by taking a square which is an aliquot part of U, as one fourth of U, and applying

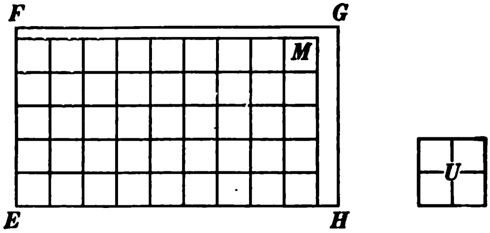


FIG. 3. Rectangle $EG = \frac{45}{4}\ U+ = 11\frac{1}{4}\ U+$.

it as a measure to the rectangle (see Fig. 3) a number will be obtained which, divided by four,* will give another (and

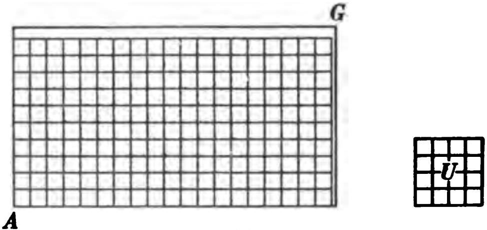


FIG. 4. Rectangle $AG = \frac{190}{16}\ U+ = 11\frac{7}{8}\ U+$.

usually closer) approximate area of the given rectangle. By proceeding in this way (see Fig. 4), closer and closer approximations of the true area may be obtained.

---

* It takes four of the small squares to make the *unit* itself.

469. If the sides of the given rectangle and of the unit square are *commensurable*, a square may be found which is an aliquot part of U, and which is contained in the rectangle also an integral number of times.

470. If the sides of the given rectangle and of the unit square are *incommensurable*, then closer and closer approximations to the area may be obtained, but no square which is an aliquot part of U will be also an aliquot part of the rectangle (by definition of incommensurable magnitudes). There is, however, a definite *limit* which is approached more and more closely by the approximations obtained by using smaller and smaller subdivisions of the unit square, as these subdivisions approach zero as a limit.*

471. Def. The **measure-number**, or **area**, of a rectangle which is incommensurable with the chosen unit square is the *limit* which successive approximate measure-numbers of the rectangle *approach* as the subdivisions of the unit square approach zero as a limit.

For brevity the expression, *the area of a figure*, is used to mean the measure-number of the surface of the figure with respect to a chosen unit.

472. Def. The **ratio** of any two surfaces is the ratio of their measure-numbers (based on the same unit).

473. Def. Equivalent figures are figures which have the same area.

The student should note that:

Equal figures have the same *shape* and *size;* such figures can be made to coincide.

Similar figures have the same *shape.*

Equivalent figures have the same *size.*

Ex. 799. Draw two equivalent figures that are not equal.

---

* For none of these approximations can exceed a certain fixed number, for example $(h+1)(b+1)$, where the measure applied is contained in the altitude h times with a remainder less than the measure, and in the base b times with a remainder less than the measure.

Ex. 800. Draw two equal figures on the blackboard or cut them out of paper, and show that equal figures may be added to them in such a way that the resulting figures are not equal. Are they the same *size*?

Ex. 801. Draw figures to show that axioms 2, 7, and 8, when applied to equal figures, do not give results which satisfy the test for equal figures.

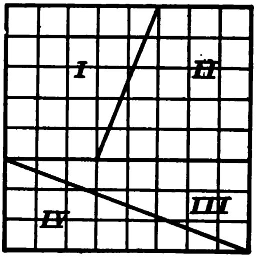


FIG. 1.

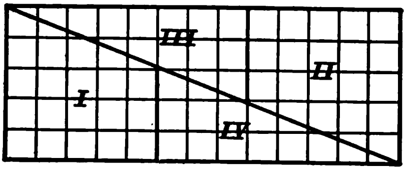


FIG. 2.

Ex. 802. Fig. 1 above represents a card containing 64 small squares, cut into four pieces, I, II, III, and IV. Fig. 2 represents these four pieces placed together in different positions forming, as it would seem, a rectangle containing 65 of these small squares. By your knowledge of similar triangles, try to explain the fallacy in the construction.

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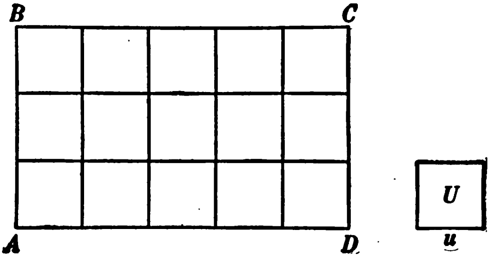
474. Historical Note. Geometry is supposed to have had its origin in land surveying, and the earliest traditions state that it had its beginning in Egypt and Babylon. The records of Babylon were made on clay tablets, and give methods for finding the approximate areas of several rectilinear figures, and also of the circle. The Egyptian records were made on papyrus. Herodotus states that the fact that the inundations of the Nile caused changes in the amount of taxable land, rendered it necessary to devise accurate land measurements.

This work was done by the Egyptian priests, and the earliest manuscript extant is that of Ahmes, who lived about 1700 B.C. This manuscript, known as the Rhind papyrus, is preserved in the British Museum. It is called "Directions for knowing all dark things," and is thought to be a copy of an older manuscript, dating about 3400 B.C. In addition to problems in arithmetic it contains a discussion of areas. Problems on pyramids follow, which show some knowledge of the properties of similar figures and of trigonometry, and which give dimensions, agreeing closely with those of the great pyramids of Egypt.

The geometry of the Egyptians was concrete and practical, unlike that of the Greeks, which was logical and deductive, even from its beginning.

PROPOSITION I. THEOREM

475. *The area of a rectangle is equal to the product of its base and its altitude.* (See § 476.)



Given rectangle *ABCD*, with base *AD* and altitude *AB*, and let *U* be the chosen unit of surface, whose side is *u*.

To prove the area of $ABCD = AD \cdot AB$.

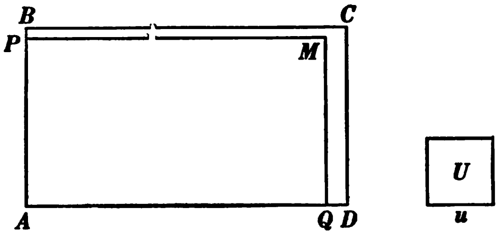
I. If *AD* and *AB* are each commensurable with *u*.

(*a*) Suppose that *u* is contained in *AD* and *AB* each an integral number of times.

ARGUMENT	REASONS
1. Lay off *u* upon *AD* and *AB*, respectively. Suppose that *u* is contained in *AD* *r* times, and in *AB* *s* times.	1. § 335.
2. At the points of division on *AD* and on *AB* erect $\perp$s to *AD* and *AB*, respectively.	2. § 63.
3. Then rectangle *ABCD* is divided into unit squares.	3. § 466.
4. There are *r* of these unit squares in a row on *AD*, and *s* rows of these squares in rectangle *ABCD*.	4. Arg. 1.
5. $\therefore$ the area of $ABCD = r \cdot s$.	5. § 467.
6. But *r* and *s* are the measure-numbers of *AD* and *AB*, respectively, referred to the linear unit *u*.	6. Arg. 1.
7. $\therefore$ the area of $ABCD = AD \cdot AB$. Q.E.D.	7. § 309.

(*b*) If u is not a measure of AD and AB, respectively, but if some aliquot part of u is such a measure.

The proof is left to the student.



II. If AD and AB are each incommensurable with u.

ARGUMENT	REASONS
1. Let m be a measure of u. Apply m as a measure to AD and AB as many times as possible. There will be a remainder, as QD, on AD, and a remainder, as PB, on AB, each less than m.	1. § 339.
2. Through Q draw $QM \perp AD$, and through P draw $PM \perp AB$.	2. § 63.
3. Now AQ and AP are each commensurable with the measure m, and hence commensurable with u.	3. § 337.
4. ∴ the area of rectangle $APMQ = AQ \cdot AP$.	4. § 475, I.
5. Now take a smaller measure of u. No matter how small a measure of u is taken, when it is applied as a measure to AD and AB, the remainders, QD and PB, will be smaller than the measure taken.	5. § 335.
6. ∴ the difference between AQ and AD may be made to become and remain less than any previously assigned segment, however small.	6. Arg. 5.

Argument	Reasons
7. Likewise the difference between AP and AB may be made to become and remain less than any previously assigned segment, however small.	7. Arg. 5.
8. ∴ AQ approaches AD as a limit, and AP approaches AB as a limit.	8. § 349.
9. ∴ $AQ \cdot AP$ approaches $AD \cdot AB$ as a limit.	9. § 477.
10. Again, the difference between $APMQ$ and $ABCD$ may be made to become and remain less than any previously assigned area, however small.	10. Arg. 5.
11. ∴ $APMQ$ approaches $ABCD$ as a limit.	11. § 349.
12. But the area of $APMQ$ is always equal to $AQ \cdot AP$.	12. Arg. 4.
13. .. the area of $ABCD = AD \cdot AB$.	13. § 355.
Q.E.D.	

III. If AD is commensurable with u but AB incommensurable with u. The proof is left as an exercise for the student

476. Note. By the product of two lines is meant the product of the *measure-numbers* of the lines. The proof that *to every straight line segment there belongs a measure-number* is given in § 595.

477. *If each of any finite number of variables approaches a finite limit, not zero, then the limit of their product is equal to the product of their limits.* (See § 593.)

478. Cor. I. *The area of a square is equal to the square of its side.*

479. Cor. II. *Any two rectangles are to each other as the products of their bases and their altitudes.*

Outline of Proof. Denote the two rectangles by R and R', their bases by b and b', and their altitudes by h and h', respectively. Then $R = b \cdot h$ and $R' = b' \cdot h'$. ∴ $\frac{R}{R'} = \frac{b \cdot h}{b' \cdot h'}$.

480. Cor. III. (*a*) *Two rectangles having equal bases are to each other as their altitudes, and* (*b*) *two rectangles having equal altitudes are to each other as their bases.*

OUTLINE OF PROOF

$$(a)\ \frac{R}{R'}=\frac{b\cdot h}{b\cdot h'}=\frac{h}{h'}. \qquad (b)\ \frac{R}{R'}=\frac{b\cdot h}{b'\cdot h}=\frac{b}{b'}.$$

Ex. 803. Draw a rectangle whose base is 7 units and whose altitude is 4 units and show how many unit squares it contains.

Ex. 804. Find the area of a rectangle whose base is 12 inches and whose altitude is 5 inches.

Ex. 805. Find the area of a rectangle whose diagonal is 10 inches, and one of whose sides is 6 inches.

Ex. 806. If the area of a rectangle is 60 square feet, and the base, 5 inches, what is the altitude?

Ex. 807. If the base and altitude of a rectangle are $2\frac{1}{4}$ inches and $1\frac{1}{4}$ inches, respectively, find the area of the rectangle.

Ex. 808. Find the area of a square whose diagonal is $8\sqrt{2}$ inches.

Ex. 809. Find the successive approximations to the area of a rectangle if its sides are $\sqrt{10}$ and $\sqrt{5}$, respectively, using 3 times 2 for the first approximation, taking the square roots to tenths for the next, to hundredths for the next, etc.

Ex. 810. Compare two rectangles if a diagonal and a side of one are d and s, respectively, while a diagonal and side of the other are d' and s'.

Ex. 811. Construct a rectangle whose area shall be three times that of a given rectangle.

Ex. 812. Construct a rectangle which shall be to a given rectangle in the ratio of two given lines, m and n.

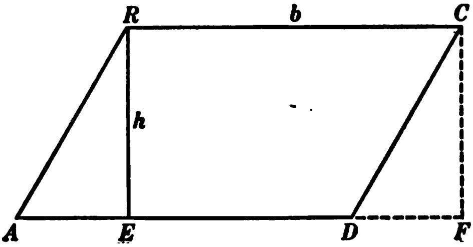
Ex. 813. Compare two rectangles whose altitudes are equal, but whose bases are 15 inches and 3 inches, respectively.

Ex. 814. From a given rectangle cut off a rectangle whose area is two thirds that of the given one.

Ex. 815. If the base and altitude of a certain rectangle are 12 inches and 8 inches, respectively, and the base and altitude of a second rectangle are 6 inches and 4 inches, respectively, compare their areas.

PROPOSITION II. THEOREM

481. *The area of a parallelogram equals the product of its base and its altitude.*



Given ▱ $ARCD$, with base b and altitude h.

To prove area of $ARCD$, $= b \cdot h$.

ARGUMENT	REASONS
1. Draw the rectangle $ERCF$, having b as base and h as altitude.	1. § 223.
2. In rt. △s DCF and ARE, $DC = AR$.	2. § 232.
3. Also $CF = RE$.	3. § 232.
4. $\therefore \triangle DCF = \triangle ARE$.	4. § 211.
5. Now figure $ARCF$ = figure $ARCF$.	5. By iden.
6. $\therefore$ area of $ARCD$ = area of $ERCF$.	6. § 54, 3.
7. But area of $ERCF = b \cdot h$.	7. § 475.
8. $\therefore$ area of $ARCD = b \cdot h$. Q.E.D.	8. § 54, 1.

482. Cor. I. *Parallelograms having equal bases and equal altitudes are equivalent.*

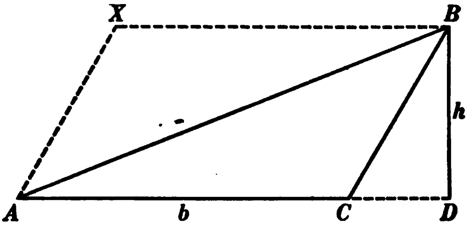
483. Cor. II. *Any two parallelograms are to each other as the products of their bases and their altitudes.*

HINT. Give a proof similar to that of § 479.

484. Cor. III. (*a*) *Two parallelograms having equal bases are to each other as their altitudes, and* (*b*) *two parallelograms having equal altitudes are to each other as their bases.* (HINT. Give a proof similar to that of § 480.)

Proposition III. Theorem

485. *The area of a triangle equals one half the product of its base and its altitude.*



Given $\triangle ABC$, with base b and altitude h.

To prove area of $\triangle ABC = \frac{1}{2} b \cdot h$.

Argument	Reasons
1. Through A draw a line $\parallel CB$, and through B draw a line $\parallel CA$. Let these lines intersect at X.	1. § 179.
2. Then $AXBC$ is a $\square$.	2. § 220.
3. $\therefore \triangle ABC = \frac{1}{2} \square AXBC$.	3. § 236.
4. But area of $AXBC = b \cdot h$.	4. § 481.
5. $\therefore$ area of $\triangle ABC = \frac{1}{2} b \cdot h$. Q.E.D.	5. § 54, 8 *a*.

486. Cor. I. *Triangles having equal bases and equal altitudes are equivalent.*

487. Cor. II. *Any two triangles are to each other as the products of their bases and their altitudes.*

Outline of Proof

Denote the two $\triangle$ by T and T', their bases by b and b', and their altitudes by h and h', respectively. Then $T = \frac{1}{2} b \cdot h$ and $T' = \frac{1}{2} b' \cdot h'$. $\therefore \frac{T}{T'} = \frac{\frac{1}{2} b \cdot h}{\frac{1}{2} b' \cdot h'} = \frac{b \cdot h}{b' \cdot h'}$.

488. Cor. III. (*a*) *Two triangles having equal bases are to each other as their altitudes, and* (*b*) *two triangles having equal altitudes are to each as their bases.*

OUTLINE OF PROOF

$$(a)\ \frac{T}{T'}=\frac{b\cdot h}{b\cdot h'}=\frac{h}{h'};\quad (b)\ \frac{T}{T'}=\frac{b\cdot h}{b'\cdot h}=\frac{b}{b'}.$$

489. Cor. IV. *A triangle is equivalent to one half of a parallelogram having the same base and altitude.*

Ex. 816. Draw four equivalent parallelograms on the same base.

Ex. 817. Find the area of a parallelogram having two sides 8 inches and 12 inches, respectively, and the included angle 60°. Find the area if the included angle is 45°.

Ex. 818. Find the ratio of two rhombuses whose perimeters are 24 inches and 16 inches, respectively, and whose smaller base angles are 30°.

Ex. 819. Find the area of an equilateral triangle having a side equal to 6 inches.

Ex. 820. Find the area of an equilateral triangle whose altitude is 8 inches.

Ex. 821. Construct three or more equivalent triangles on the same base.

Ex. 822. Find the locus of the vertices of all triangles equivalent to a given triangle and standing on the same base.

Ex. 823. Construct a triangle equivalent to a given triangle and having one of its sides equal to a given line.

Ex. 824. Construct a triangle equivalent to a given triangle and having one of its angles equal to a given angle.

Ex. 825. Construct a triangle equivalent to a given triangle and having two of its sides equal, respectively, to two given lines.

Ex. 826. Divide a triangle into three equivalent triangles by drawing lines through one of its vertices.

Ex. 827. Construct a triangle equivalent to $\frac{2}{3}$ of a given triangle; $\frac{3}{5}$ of a given triangle.

Ex. 828. Construct a triangle equivalent to a given square.

Ex. 829. The area of a rhombus is equal to one half the product of its diagonals.

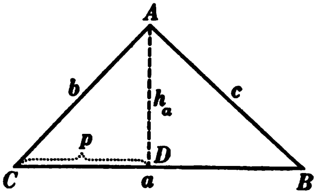
Ex. 830. If from any point in a diagonal of a parallelogram lines are drawn to the opposite vertices, two pairs of equivalent triangles are formed.

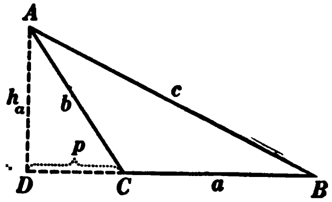
Ex. 831. Two lines joining the mid-point of the diagonal of a quadrilateral to the opposite vertices divide the figure into two equivalent parts.

Ex. 832. Find the area of a triangle if two of its sides are 6 inches and 9 inches, respectively, and the included angle is 60°.

PROPOSITION IV. PROBLEM

490. *To derive a formula for the area of a triangle in terms of its sides.*





Given △ *ABC*, with sides a, b, and c.

To derive a formula for the area of △ *ABC* in terms of a, b, and c.

ARGUMENT	REASONS
1. Let h_a denote the altitude upon a, p the projection of b upon a, and T the area of △ *ABC*. Then $T = \frac{1}{2} a h_a = \frac{a}{2} h_a$.	1. § 485.
2. $h_a^2 = b^2 - p^2 = (b+p)(b-p)$.	2. § 447.
3. But $p = \frac{a^2+b^2-c^2}{2a}$ (Fig. 1), or $-\frac{a^2+b^2-c^2}{2a}$ (Fig. 2).	3. § 456.
4. $\therefore h_a^2 = \left(b + \frac{a^2+b^2-c^2}{2a}\right)\left(b - \frac{a^2+b^2-c^2}{2a}\right)$ $= \frac{2ab+a^2+b^2-c^2}{2a} \cdot \frac{2ab-a^2-b^2+c^2}{2a}$ $= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}$.	4. § 309.
5. $\therefore h_a = \sqrt{\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}}$.	5. § 54, 13.
6. Now let $a+b+c = 2s$. Then $a+b-c = 2s-2c = 2(s-c)$; $a-b+c = 2s-2b = 2(s-b)$; $b+c-a = 2s-2a = 2(s-a)$.	6. § 54, 3.

ARGUMENT	REASONS
7. Then $h_a = \sqrt{\dfrac{2s\,2(s-a)\,2(s-b)\,2(s-c)}{4a^2}}$	7. § 309.
$= \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$	
8. $\therefore T = \dfrac{a}{2}\cdot\dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$	8. § 309.
$= \sqrt{s(s-a)(s-b)(s-c)}.$ Q.E.F.	

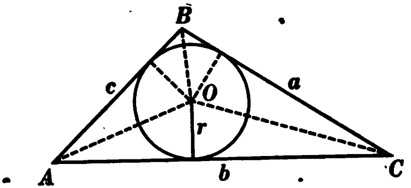
Ex. 833. Find the area of a triangle whose sides are 7, 10, and 13.

Ex. 834. If the sides of a triangle are a, b, and c, write the formula for the altitude upon b; upon c. (See Prop. IV, Arg. 7.)

Ex. 835. In triangle ABC, $a=8$, $b=12$, $c=16$; find the area of triangle ABC; the altitude upon b; the altitude upon c.

PROPOSITION V. THEOREM

491. *The area of a triangle is equal to one half the product of its perimeter and the radius of the inscribed circle.*



Given $\triangle ABC$, with area T, sides a, b, and c, and radius of inscribed circle r.

To prove $T = \frac{1}{2}(a+b+c)r$.

OUTLINE OF PROOF

1. Area of $\triangle OBC = \frac{1}{2}a\cdot r$; area of $\triangle OCA = \frac{1}{2}b\cdot r$; area of $\triangle OAB = \frac{1}{2}c\cdot r$. 2. $\therefore T = \frac{1}{2}(a+b+c)r$. Q.E.D.

492. Cor. *The area of any polygon circumscribed about a circle is equal to one half its perimeter multiplied by the radius of the inscribed circle.*

Ex. 836. If the area of a triangle is $\frac{15}{4}\sqrt{3}$ square inches and its sides are 3, 5, and 7 inches, find the radius of the inscribed circle.

Ex. 837. Derive a formula for the radius of a circle inscribed in a triangle in terms of the sides of the triangle.

Outline of Solution

$$1. \quad T = \tfrac{1}{2}(a+b+c)r = \tfrac{1}{2}(2s)r = sr.$$

$$2. \quad \therefore r = \frac{T}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}. \qquad \text{Q.E.F.}$$

Ex. 838. Derive a formula for the radius of a circle circumscribed about a triangle, in terms of the sides of the triangle.

Outline of Solution (See figure for Ex. 798.)

$$1. \quad dh = ac; \text{ i.e. } d \text{ or } 2R = \frac{ac}{h}.$$

$$2. \quad \therefore R = \frac{ac}{2h} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad \text{Q.E.F.}$$

Ex. 839. If the sides of a triangle are 9, 10, and 11, find the radius of the inscribed circle; the radius of the circumscribed circle.

Ex. 840. The sides of a triangle are $3a$, $4a$, and $5a$. Find the radius of the inscribed circle; the radius of the circumscribed circle. What kind of a triangle is it? Verify your answer by comparing the radius of the circumscribed circle with the longest side.

Ex. 841. Derive formulas for the bisectors of the angles of a triangle in terms of the sides of the triangle.

Outline of Solution (See figure for Ex. 797.)

1. $t_b^2 = ac - rs$. 2. But $a : c = r : s$. 3. $\therefore a + c : a = b : r$.

4. $\therefore r = \dfrac{ab}{a+c}$. 5. Likewise $s = \dfrac{bc}{a+c}$. 6. $\therefore t_b^2 = ac - \dfrac{ab^2c}{(a+c)^2} = \dfrac{ac(a+b+c)(a-b+c)}{(a+c)^2} = \dfrac{4acs(s-b)}{(a+c)^2}$. 7. $\therefore t_b = \dfrac{2}{a+c}\sqrt{acs(s-b)}$.

8. Likewise $t_a = \dfrac{2}{b+c}\sqrt{bcs(s-a)}$ and $t_c = \dfrac{2}{a+b}\sqrt{abs(s-c)}$. Q.E.F.

Find r, R, T, h_a, m_a, and t_a, having given:

Ex. 842. $a = 11$, $b = 9$, $c = 16$. What kind of an angle is C?

Ex. 843. $a = 13$, $b = 15$, $c = 20$. What kind of an angle is C?

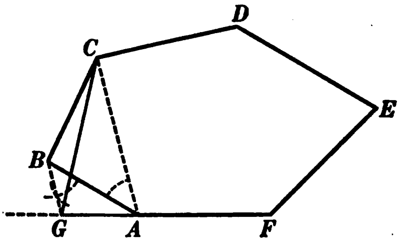
Ex. 844. $a = 24$, $b = 10$, $c = 26$. What kind of an angle is C?

TRANSFORMATION OF FIGURES

493. Def. To **transform** a figure means to find another figure which is equivalent to it.

Proposition VI. Problem

494. *To construct a triangle equivalent to a given polygon.*



Given polygon *ABCDEF*.

To construct a $\triangle \Leftrightarrow$ polygon *ABCDEF*.

(*a*) Construct a polygon $\Leftrightarrow$ *ABCDEF*, but having one side less.

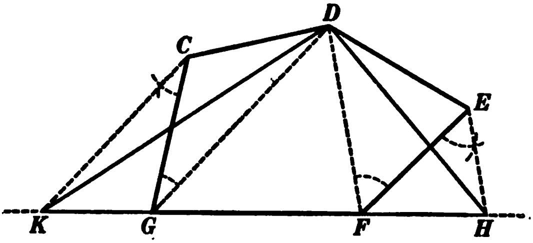
I. Construction

1. Join any two alternate vertices, as *C* and *A*.
2. Construct $BG \parallel CA$, meeting *FA* prolonged at *G*. § 188.
3. Draw *CG*.
4. Polygon *GCDEF* $\Leftrightarrow$ polygon *ABCDEF* and has one side less.

II. Proof

Argument	Reasons
1. $\triangle$s *AGC* and *ABC* have the same base *CA*, and the same altitude, the $\perp$ between the $\parallel$s *CA* and *BG*.	1. § 235.
2. $\therefore \triangle AGC \Leftrightarrow \triangle ABC$.	2. § 486.
3. But polygon *ACDEF* = polygon *ACDEF*.	3. By iden.
4. $\therefore$ polygon *GCDEF* $\Leftrightarrow$ polygon *ABCDEF*.	4. § 54, 2.

Q.E.D.



(*b*) In like manner, reduce the number of sides of the new polygon *GCDEF* until △ *DHK* is obtained.

The construction, proof, and discussion are left as an exercise for the student.

Ex. 845. Transform a scalene triangle into an isosceles triangle.

Ex. 846. Transform a trapezoid into a right triangle.

Ex. 847. Transform a parallelogram into a trapezoid.

Ex. 848. Transform a pentagon into an isosceles triangle.

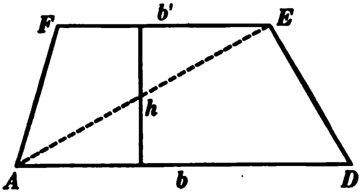
Ex. 849. Construct a triangle equivalent to $\frac{3}{5}$ of a given trapezium.

Ex. 850. Transform $\frac{4}{7}$ of a given pentagon into a triangle.

Ex. 851. Construct a rhomboid and a rhombus which are equivalent, and which have a common diagonal.

PROPOSITION VII. THEOREM

495. *The area of a trapezoid equals the product of its altitude and one half the sum of its bases.*



Given trapezoid *AFED*, with altitude h and bases b and b'.

To prove area of $AFED = \frac{1}{2}(b + b')h$.

ARGUMENT	REASONS
1. Draw the diagonal AE.	1. § 54, 15.
2. The altitude of $\triangle AED$, considering b as base, is equal to the altitude of $\triangle AFE$, considering b' as base, each being equal to the altitude of the trapezoid, h.	2. § 235.
3. $\therefore$ area of $\triangle AED = \frac{1}{2} b \cdot h$.	3. § 485.
4. Area of $\triangle AFE = \frac{1}{2} b' \cdot h$.	4. § 485.
5. $\therefore$ area of trapezoid $AFED = \frac{1}{2}(b + b')h$.	5. § 54, 2.

Q.E.D.

496. Cor. *The area of a trapezoid equals the product of its altitude and its median.*

497. Question. The ancient Egyptians, in attempting to find the area of a field in the shape of a trapezoid, multiplied one half the sum of the parallel sides by one of the other sides. For what figure would this method be correct?

Ex. 852. Find the area of a trapezoid whose bases are 7 inches and 9 inches, respectively, and whose altitude is 5 inches.

Ex. 853. Find the area of a trapezoid whose median is 10 inches and whose altitude is 6 inches.

Ex. 854. Through a given point in one side of a given parallelogram draw a line which shall divide the parallelogram into two equivalent parts. Will these parts be equal?

Ex. 855. Through a given point within a parallelogram draw a line which shall divide the parallelogram into two equivalent parts. Will these parts be equal?

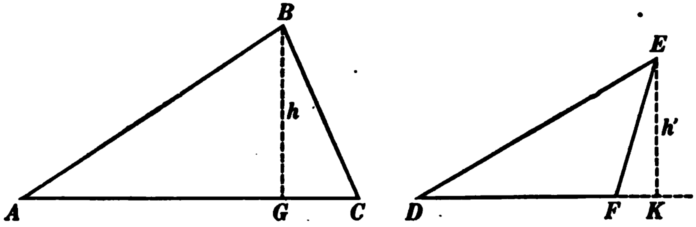
Ex. 856. If the mid-point of one of the non-parallel sides of a trapezoid is joined to the extremities of the other of the non-parallel sides, the area of the triangle formed is equal to one half the area of the trapezoid.

Ex. 857. Find the area of a trapezoid whose bases are b and b' and whose other sides are each equal to s.

Ex. 858. If the sides of any quadrilateral are bisected and the points of bisection joined, the included figure will be a parallelogram equal in area to half the original figure.

PROPOSITION VIII. THEOREM

498. *Two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Given $\triangle ABC$ and DEF, with $\angle A = \angle D$.

To prove $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AC \cdot AB}{DF \cdot DE}$.

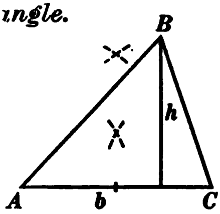
ARGUMENT	REASONS
1. Let h be the altitude of $\triangle ABC$ upon side AC, and h' the altitude of $\triangle DEF$ upon side DF. Then, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AC \cdot h}{DF \cdot h'} = \dfrac{AC}{DF} \cdot \dfrac{h}{h'}$.	1. § 487.
2. In rt. $\triangle ABG$ and DEK, $\angle A = \angle D$.	2. By hyp.
3. $\therefore \triangle ABG \sim \triangle DEK$.	3. § 422.
4. $\therefore \dfrac{h}{h'} = \dfrac{AB}{DE}$.	4. § 424, 2.
5. $\therefore \dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AC}{DF} \cdot \dfrac{AB}{DE} = \dfrac{AC \cdot AB}{DF \cdot DE}$. Q.E.D.	5. § 309.

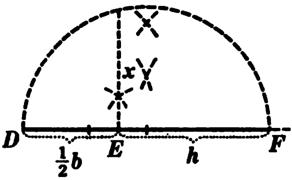
Ex. 859. Draw two triangles upon the blackboard, so that an angle of one shall equal an angle of the other. Give a rough estimate in inches of the sides including the equal angles in the two triangles, and compute the numerical ratio of the triangles.

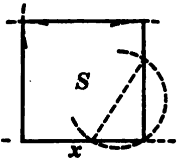
Ex. 860. If two triangles have an angle of one supplementary to an angle of the other, the triangles are to each other as the products of the sides including the supplementary angles.

PROPOSITION IX. PROBLEM

499. *To construct a square equivalent to a given triangle.*







Given $\triangle ABC$, with base b and altitude h.

To construct a square $\Bumpeq \triangle ABC$.

I. Analysis

1. Let $x =$ the side of the required square; then $x^2 =$ area of required square.

2. $\frac{1}{2}b \cdot h =$ area of the given $\triangle ABC$.

3. $\therefore x^2 = \frac{1}{2}b \cdot h$.

4. $\therefore \frac{1}{2}b : x = x : h$.

5. $\therefore$ the side of the required square will be a mean proportional between $\frac{1}{2}b$ and h.

II. Construction

1. Construct a mean proportional between $\frac{1}{2}b$ and h. Call it x. § 445.

2. On x, as base, construct a square, S.

3. S is the required square.

III. Proof

ARGUMENT	REASONS
1. $\frac{1}{2}b : x = x : h$.	1. By cons.
2. $\therefore x^2 = \frac{1}{2}bh$.	2. § 388.
3. But $x^2 =$ area of S.	3. § 478.
4. And $\frac{1}{2}b \cdot h =$ area of $\triangle ABC$.	4. § 485.
5. $\therefore S \Bumpeq \triangle ABC$. Q.E.D.	5. § 54, 1.

IV. The discussion is left as an exercise for the student.

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500. Question. Could x be constructed as a mean proportional between b and $\frac{1}{2}h$?

501. Problem. *To construct a square equivalent to a given parallelogram.*

Ex. 861. Construct a square equivalent to a given rectangle.

Ex. 862. Construct a square equivalent to a given trapezoid.

Ex. 863. Upon a given base construct a triangle equivalent to a given parallelogram.

Ex. 864. Construct a rectangle having a given base and equivalent to a given square.

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502. Props. VI, VIII, and IX form the basis of a large class of important constructions.

(*a*) Prop. VI enables us to construct a triangle equivalent to any polygon. It is then an easy matter to construct a trapezoid, an isosceles trapezoid, a parallelogram, a rectangle, or a rhombus equivalent to the triangle and hence equivalent to the given polygon.

(*b*) Prop. VIII gives us a method for constructing an equilateral triangle equivalent to any given triangle. (See Ex. 865.) Hence Prop. VIII, with Prop. VI, enables us to construct an equilateral triangle equivalent to any given polygon.

(*c*) Likewise Prop. IX, with Prop. VI, enables us to construct a square equivalent to any given polygon or to any fractional part or to any multiple of any given polygon.

Ex. 865. (*a*) Transform triangle ABC into triangle DBC, retaining base BC and making angle $DBC = 60°$.

(*b*) Transform triangle DBC into triangle EBF, retaining angle $DBC = 60°$ and making sides EB and BF equal. (Each will be a mean proportional between DB and BC.)

(*c*) What kind of a triangle is EBF?

Ex. 866. Transform a parallelogram into an equilateral triangle.

Ex. 867. Construct an equilateral triangle equivalent to $\frac{2}{3}$ of a given trapezium.

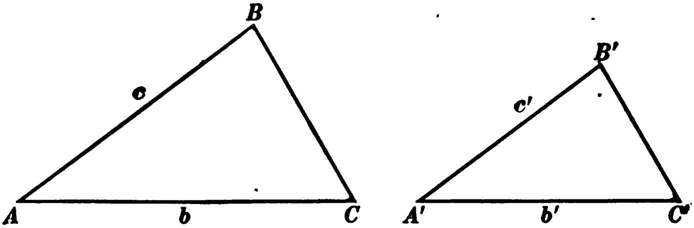
Ex. 868. Construct each of the following figures equivalent to $\frac{3}{5}$ of a given irregular pentagon: (1) a triangle; (2) an isosceles triangle; (3) a right triangle; (4) an equilateral triangle; (5) a trapezium; (6) a trapezoid; (7) an isosceles trapezoid; (8) a parallelogram; (9) a rhombus; (10) a rectangle; (11) a square.

Ex. 869. Transform a trapezoid into a right triangle having the hypotenuse equal to a given line. What restrictions are there upon the given line?

Ex. 870. Construct a triangle equivalent to a given trapezoid, and having a given line as base and a given angle adjacent to the base.

PROPOSITION X. THEOREM

503. *Two similar triangles are to each other as the squares of any two homologous sides.*



Given two similar $\triangle$ ABC and $A'B'C'$, with b and b' two homol. sides.

To prove $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{b^2}{b'^2}$.

ARGUMENT	REASONS
1. $\triangle ABC \sim \triangle A'B'C'$.	1. By hyp.
2. $\therefore \angle A = \angle A'$.	2. § 424, 1.
3. Then $\dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{b \cdot c}{b' \cdot c'} = \dfrac{b}{b'} \cdot \dfrac{c}{c'}$.	3. § 498.
4. But $\dfrac{c}{c'} = \dfrac{b}{b'}$.	4. § 424, 2.
5. $\therefore \dfrac{\triangle ABC}{\triangle A'B'C'} = \dfrac{b}{b'} \cdot \dfrac{b}{b'} = \dfrac{b^2}{b'^2}$. Q.E.D.	5. § 309.

504. Cor. *Two similar triangles are to each other as the squares of any two homologous altitudes.* (See § 435.)

Ex. 871. Two similar triangles are to each other as the squares of two homologous medians.

Ex. 872. Construct a triangle similar to a given triangle and having an area four times as great.

Ex. 873. Construct a triangle similar to a given triangle and having an area twice as great.

Ex. 874. Divide a given triangle into two equivalent parts by a line parallel to the base.

Ex. 875. Prove Prop. X by using § 487.

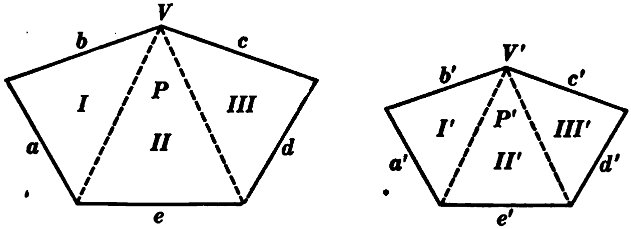
Ex. 876. Draw a line parallel to the base of a triangle and cutting off a triangle that shall be equivalent to one third of the given triangle.

Ex. 877. In two similar triangles a pair of homologous sides are 10 feet and 6 feet, respectively. Find the homologous side of a similar triangle equivalent to their difference.

Ex. 878. Construct an equilateral triangle whose area shall be three fourths that of a given square.

PROPOSITION XI. THEOREM

505. *Two similar polygons are to each other as the squares of any two homologous sides.*



Given two similar polygons P and P' in which a and a', b and b', etc., are pairs of homol. sides.

To prove $\frac{P}{P'} = \frac{a^2}{a'^2}$

Argument	**Reasons**
1. Draw all possible diagonals from any two homol. vertices, as V and V'.	1. § 54, 15.
2. Then the polygons will be divided into the same number of $\triangle \sim$ each to each and similarly placed, as $\triangle$ I and $\triangle$ I′, $\triangle$ II and $\triangle$ II′, etc.	2. § 439.
3. Then $\frac{\triangle \text{I}}{\triangle \text{I}'} = \frac{a^2}{a'^2}$.	3. § 503.
4. $\frac{\triangle \text{II}}{\triangle \text{II}'} = \frac{e^2}{e'^2}$.	4. § 503.
5. $\frac{\triangle \text{III}}{\triangle \text{III}'} = \frac{d^2}{d'^2}$.	5. § 503.
6. But $\frac{a}{a'} = \frac{e}{e'} = \frac{d}{d'}$.	6. § 419.
7. $\therefore \frac{a^2}{a'^2} = \frac{e^2}{e'^2} = \frac{d^2}{d'^2}$.	7. § 54, 13.
8. $\therefore \frac{\triangle \text{I}}{\triangle \text{I}'} = \frac{\triangle \text{II}}{\triangle \text{II}'} = \frac{\triangle \text{III}}{\triangle \text{III}'}$.	8. § 54, 1.
9. $\therefore \frac{\triangle \text{I} + \triangle \text{II} + \triangle \text{III}}{\triangle \text{I}' + \triangle \text{II}' + \triangle \text{III}'} = \frac{\triangle \text{I}}{\triangle \text{I}'} = \frac{a^2}{a'^2}$.	9. § 401.
10. $\therefore \frac{P}{P'} = \frac{a^2}{a'^2}$. Q.E.D.	10 § 309.

506. **Cor.** *Two similar polygons are to each other as the squares of any two homologous diagonals.*

Ex. 879. If one square is double another, what is the ratio of their sides?

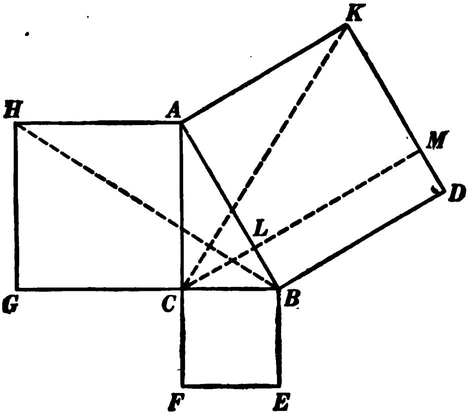
Ex. 880. Divide a given hexagon into two equivalent parts so that one part shall be a hexagon similar to the given hexagon.

Ex. 881. The areas of two similar rhombuses are to each other as the squares of their homologous diagonals.

Ex. 882. One side of a polygon is 8 and its area is 120. The homologous side of a similar polygon is 12; find its area.

PROPOSITION XII. THEOREM

507. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares described on the other two sides.*



Given rt. $\triangle ABC$, right-angled at C, and the squares described on its three sides.

To prove square AD ≎ square BF + square CH.

ARGUMENT	REASONS
1. From C draw $CM \perp AB$, cutting AB at L and KD at M.	1. § 155.
2. Draw CK and BH.	2. § 54, 15.
3. $\angle$s ACG, BCA, and FCB are all rt. $\angle$s.	3. By hyp.
4. $\therefore$ ACF and GCB are str. lines.	4. § 76.
5. In $\triangle$s CAK and HAB, $CA = HA$, $AK = AB$.	5. § 233.
6. $\angle CAB = \angle CAB$.	6. By iden.
7. $\angle BAK = \angle HAC$.	7. § 64.
8. $\therefore \angle CAK = \angle HAB$.	8. § 54, 2
9. $\therefore \triangle CAK = \triangle HAB$.	9. § 107.
10. $\triangle CAK$ and rectangle AM have the same base AK and the same altitude, the $\perp$ between the ‖s AK and CM.	10. § 235.

ARGUMENT	REASONS
11. ∴ $\triangle CAK \Bumpeq \frac{1}{2}$ rectangle AM.	11. § 489.
12. Likewise $\triangle HAB$ and square CH have the same base HA and the same altitude, the ⊥ between ∥s HA and GB.	12. § 235.
13. ∴ $\triangle HAB \Bumpeq \frac{1}{2}$ square CH.	13. § 489.
14. But $\triangle CAK = \triangle HAB$.	14. Arg. 9.
15. ∴ $\frac{1}{2}$ rectangle $AM \Bumpeq \frac{1}{2}$ square CH.	15. § 54, 1.
16. ∴ rectangle $AM \Bumpeq$ square CH.	16. § 54, 7 *a*.
17. Likewise, by drawing CD and AE, it may be proved that rectangle $LD \Bumpeq$ square BF.	17. By steps similar to 5–16.
18. ∴ rectangle $AM +$ rectangle $LD \Bumpeq$ square $CH +$ square BF.	18. § 54, 2.
19. ∴ square $AD \Bumpeq$ square $CH +$ square BF. Q.E.D.	19. § 309.

508. Cor. I. *The square described on either side of a right triangle is equivalent to the square described on the hypotenuse minus the square described on the other side.*

509. Cor. II. *If similar polygons are described on the three sides of a right triangle as homologous sides, the polygon described on the hypotenuse is equivalent to the sum of the polygons described on the other two sides.*

Given rt. $\triangle ABC$, right-angled at C, and let P, Q, and R be $\sim$ polygons described on a, b, and c, respectively, as homol. sides.

To prove $R \Bumpeq P + Q$.

ARGUMENT ONLY

1. $\frac{P}{R}=\frac{a^2}{c^2}$. 2. $\frac{Q}{R}=\frac{b^2}{c^2}$.

3. $\therefore \frac{P+Q}{R}=\frac{a^2+b^2}{c^2}=\frac{c^2}{c^2}=1$. 4. $\therefore R \Bumpeq P+Q$. Q.E.D.

Ex. 883. The square on the hypotenuse of an isosceles right triangle is equivalent to four times the triangle.

510. Historical Note. Prop. XII is usually known as the Pythagorean Proposition, because it was discovered by Pythagoras. The proof given here is that of Euclid (about 300 B.C.).

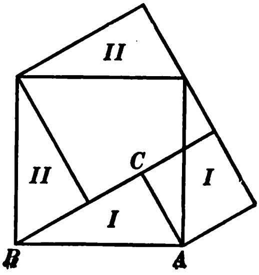
PYTHAGORAS

Pythagoras (569–500 B.C.), one of the most famous mathematicians of antiquity, was born at Samos. He spent his early years of manhood studying under Thales and traveled in Asia Minor and Egypt and probably also in Babylon and India. He returned to Samos where he established a school that was not a great success. Later he went to Crotona in Southern Italy and there gained many adherents. He formed, with his closest followers, a secret society, the members of which possessed all things in common. They used as their badge the five-pointed star or pentagram which they knew how to construct and which they considered symbolical of health. They ate simple food and practiced severe discipline, having obedience, temperance, and purity as their ideals. The brotherhood regarded their leader with reverent esteem and attributed to him their most important discoveries, many of which were kept secret.

Pythagoras knew something of incommensurable numbers and proved that the diagonal and the side of a square are incommensurable.

The first man who propounded a theory of incommensurables is said to have suffered shipwreck on account of the sacrilege, since such numbers were thought to be symbolical of the Deity.

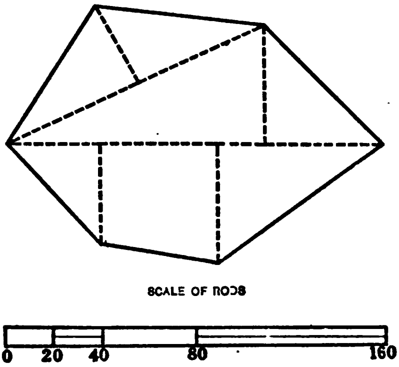
Pythagoras, having incurred the hatred of his political opponents, was murdered by them, but his school was reëstablished after his death and it flourished for over a hundred years.



Ex. 884. Use the adjoining figure to prove the Pythagorean theorem.

Ex. 885. Construct a triangle equivalent to the sum of two given triangles.

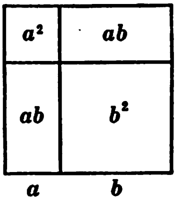
Ex. 886. The figure represents a farm drawn to the scale indicated. Make accurate measurements and calculate approximately the number of acres in the farm.



Ex. 887. A farm $XYZW$, in the form of a trapezium, has the following dimensions: $XY = 60$ rods, $YZ = 70$ rods, $ZW = 90$ rods, $WX = 100$ rods, and $XZ = 66$ rods. Draw a plot of the farm to the scale 1 inch = 40 rods, and calculate the area of the farm in acres.

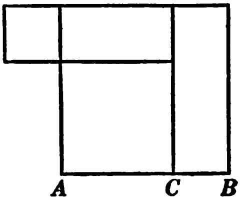
511. Def. By the **rectangle of two lines** is meant the rectangle having these two lines as adjacent sides.

Ex. 888. The square described on the sum of two lines is equivalent to the sum of the squares described on the lines plus twice their rectangle.



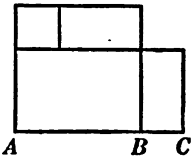
Ex. 889. The square described on the difference of two lines is equivalent to the sum of the squares described on the lines diminished by twice their rectangle.

HINT. Let AB and CB be the given lines.



Ex. 890. The rectangle whose sides are the sum and difference respectively of two lines is equivalent to the difference of the squares described on the lines.

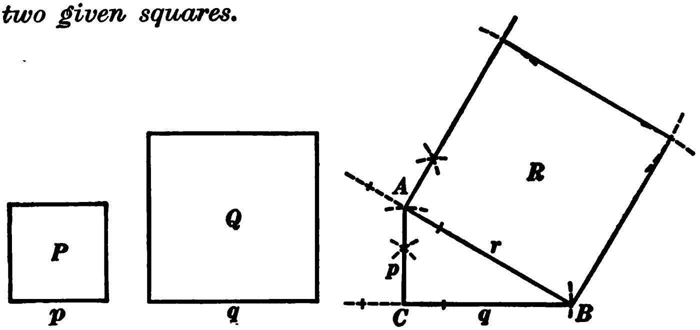
HINT. Let AB and BC be the given lines.



Ex. 891. Write the three algebraic formulas corresponding to the last three exercises.

PROPOSITION XIII. PROBLEM

512. *To construct a square equivalent to the sum of two given squares.*



Given squares P and Q.

To construct a square ≎ the sum of P and Q.

I. Construction

1. Construct the rt. △ ABC, having for its sides p and q, the sides of the given squares.
2. On r, the hypotenuse of the △, construct the square R.
3. R is the required square.

II. The proof and discussion are left to the student.

Ex. 892. Construct a square equivalent to the sum of three or more given squares.

Ex. 893. Construct a square equivalent to the difference of two squares.

Ex. 894. Construct a square equivalent to the sum of a given square and a given triangle.

Ex. 895. Construct a polygon similar to two given similar polygons and equivalent to their sum. (See § 509.)

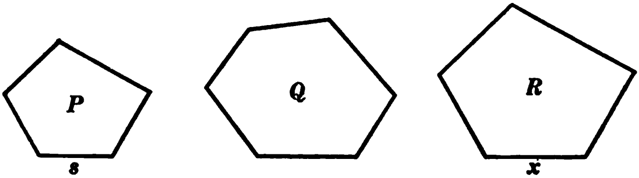
Ex. 896. Construct a polygon similar to two given similar polygons and equivalent to their difference.

Ex. 897. Construct an equilateral triangle equivalent to the sum of two given equilateral triangles.

Ex. 898. Construct an equilateral triangle equivalent to the difference of two given equilateral triangles.

PROPOSITION XIV. PROBLEM

513. *To construct a polygon similar to one of two given polygons and equivalent to the other.*



Given polygons P and Q, with s a side of P.

To construct a polygon $\sim P$ and $\Leftrightarrow Q$.

I. Analysis

1. Imagine the problem solved and let R be the required polygon with side x homol. to s, a side of P.

2. Then $P : R = s^2 : x^2$; i.e. $P : Q = s^2 : x^2$, since $Q \Leftrightarrow R$. (1)

3. Now to avoid comparing polygons which are not similar, we may reduce P and Q to $\Leftrightarrow$ squares. Let the sides of these squares be m and n, respectively; then $m^2 \Leftrightarrow P$ and $n^2 \Leftrightarrow Q$.

4. $\therefore m^2 : n^2 = s^2 : x^2$, from (1).

5. $\therefore m : n = s : x$.

6. That is, x is the fourth proportional to m, n, and s.

II. The construction, proof, and discussion are left as an exercise for the student.

514. Historical Note. This problem was first solved by Pythagoras about 550 B.C.

Ex. 899. Construct a triangle similar to a given triangle and equivalent to a given parallelogram.

Ex. 900. Construct a square equivalent to a given pentagon.

Ex. 901. Construct a triangle, given its angles and its area (equal to that of a given parallelogram). HINT. See Prop. XIV.

Ex. 902. Divide a triangle into two equivalent parts by a line drawn perpendicular to the base. HINT. Draw a median to the base, then apply Prop. XIV.

Ex. 903. Fig. 1 represents maps of Utah and Colorado drawn to the scale indicated. By carefully measuring the maps: (1) Calculate the perimeter of each state. (2) Calculate the area of each state. (3) Check your results for (2) by comparing with the areas given for these states in your geography.

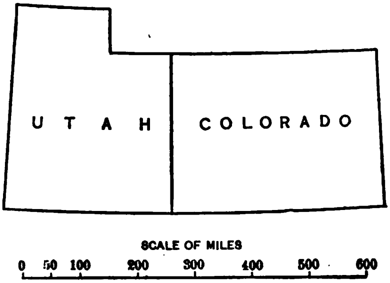


FIG. 1.

Ex. 904. Fig. 2 represents a map of Pennsylvania. A straight line from the southwest corner to the northeast corner is 300 miles long.

(1) Determine the scale to which the map is drawn.

(2) Calculate the distance from Pittsburg to Harrisburg; from Harrisburg to Philadelphia; from Philadelphia to Scranton; from Scranton to Harrisburg.

(3) Calculate approximately the area of the state in square miles.

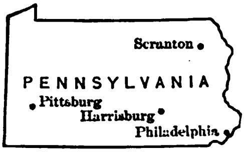


FIG. 2.

Ex. 905. Let C represent a candle; A a screen 1 foot square and 1 yard from C; B a screen 2 feet square and 2 yards from C; D a screen 3 feet square and 3 yards from C. If screen A were removed, the quantity of light it received would fall on B. What would happen if B were removed? On which screen, then, would the light be the least intense? From the figure, determine the *law of intensity of light*.

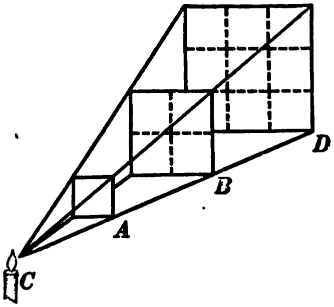


FIG. 3.

Ex. 906. Divide a triangle into two equivalent parts by a line drawn from a given point in one of its sides.

HINT. Let M be the given point in AC of $\triangle ABC$. Then, if $AM > MC$, $\frac{1}{2} AB \cdot AC = AX \cdot AM$: if $MC > AM$, $\frac{1}{2} CA \cdot CB = CM \cdot CX$, where MX is the required line.

Ex. 907. A represents a station. Cars approach the station on track BA and leave the station on track AC. Construct an arc of a circle DE, with given radius r, connecting the two intersecting car lines, and so that each car line is tangent to the arc.

This same principle is involved in designing a building between two streets forming at their point of intersection a small acute angle, as the Flatiron Building in New York City.

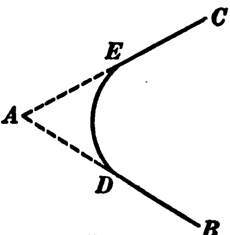


FIG. 4.

Ex. 908. Find the area of a rhombus if its diagonals are in the ratio of 5 to 7 and their sum is 16.

MISCELLANEOUS EXERCISES

Ex. 909. Show that if a and b are two sides of a triangle, the area is $\frac{1}{4}ab$ when the included angle is 30° or 150°; $\frac{1}{4}ab\sqrt{2}$ when the included angle is 45° or 135°; $\frac{1}{4}ab\sqrt{3}$ when the included angle is 60° or 120°.

Ex. 910. The sum of the perpendiculars from any point within a regular convex polygon upon the sides is constant.

HINT. Join the point with the vertices of the polygon, and consider the sum of the areas of the triangles.

Ex. 911. The sum of the squares on the segments of two perpendicular chords in a circle is equivalent to the square on the diameter.

Ex. 912. The hypotenuse of a right triangle is 20, and the projection of one arm upon the hypotenuse is 4. What is its area?

Ex. 913. A quadrilateral is equivalent to a triangle if its diagonals and the angle included between them are respectively equal to two sides and the included angle of the triangle.

Ex. 914. Transform a given triangle into another triangle containing two given angles.

Ex. 915. Prove geometrically the algebraic formula $(a+b)(c+d) = ac+bc+ad+bd$.

Ex. 916. If in any triangle an angle is equal to two thirds of a straight angle (§ 69), then the square on the side opposite is equivalent to the sum of the squares on the other two sides and the rectangle contained by them.

Ex. 917. The two medians RK and SH of the triangle RST intersect at P. Prove that the triangle RPS is equivalent to the quadrilateral $HPKT$.

Ex. 918. Find the area of a triangle if two of its sides are 6 inches and 7 inches and the included angle is 30°.

Ex. 919. By two different methods find the area of an equilateral triangle whose side is 10 inches.

Ex. 920. The area of an equilateral triangle is $36\sqrt{3}$; find a side and an altitude.

Ex. 921. By using the formula of Prop. IV, Arg. 8, derive the formula for the area of an equilateral triangle whose side is a.

Ex. 922. What does the formula for T in Prop. IV become if angle C is a right angle?

Ex. 923. Given an equilateral triangle ABC, inscribed in a circle whose center is O. At the vertex C erect a perpendicular to BC cutting the circumference at D. Draw the radii OD and OC. Prove that the triangle ODC is equilateral.

Ex. 924. Assuming that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles, prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

Ex. 925. A rhombus and a square have equal perimeters, and the altitude of the rhombus is three fourths its side; compare the areas of the two figures.

Ex. 926. The length of a chord is 10 feet, and the greatest perpendicular from the subtending arc to the chord is 2 feet 7½ inches. Find the radius of the circle.

Ex. 927. In any right triangle a line from the vertex of the right angle perpendicular to the hypotenuse divides the given triangle into two triangles similar to each other and similar to the given triangle.

Ex. 928. The bases of a trapezoid are 16 feet and 10 feet, respectively, and each of the non-parallel sides is 5 feet. Find the area of the trapezoid. Also find the area of a similar trapezoid, if each of its non-parallel sides is 3 feet.

Ex. 929. A triangle having a base of 8 inches is cut by a line parallel to the base and 6 inches from it. If the base of the smaller triangle thus formed is 5 inches, find the area of the larger triangle.

Ex. 930. If the ratio of similitude of two similar triangles is 7 to 1, how often is the less contained in the greater? Hint. See §§ 418, 503.

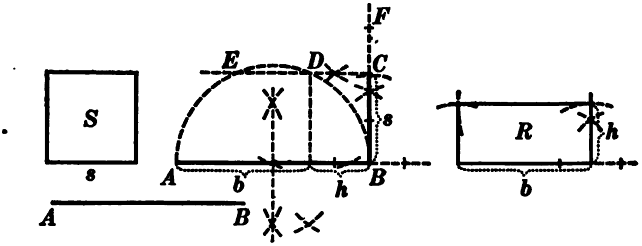
Ex. 931. Construct a square equivalent to one third of a given square.

Ex. 932. If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?

Ex. 933. Divide a right triangle into two isosceles triangles.

EXERCISES OF GREATER DIFFICULTY

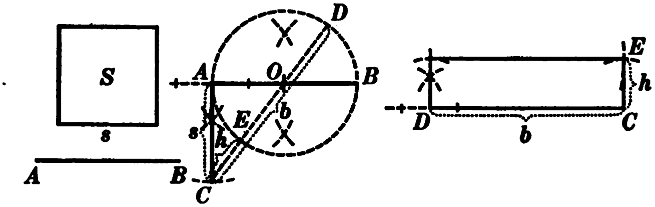
Ex. 934. Construct a rectangle equivalent to a given square and having the sum of its base and altitude equal to a given line.



Ex. 935. Construct a rectangle equivalent to a given triangle and having its perimeter equal to a given line.

Ex. 936. Construct two lines, having given their sum and their product.

Ex. 937. Construct a rectangle equivalent to a given square and having the difference of its base and altitude equal to a given line.



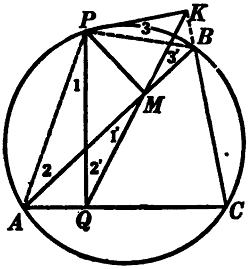
Ex. 938. Construct a rectangle equivalent to a given rhombus, the difference of the base and altitude of the rectangle being equal to a given line.

Ex. 939. Construct two lines, having given their difference and their product.

Ex. 940. Construct a triangle, given its three altitudes. Hint. Construct first an auxiliary triangle, with sides x, y, and z, such that $h_a : h_b = y : x$ and $h_c : h_a = x : z$. (See Ex. 665.)

Ex. 941. Through the vertices of an equilateral triangle draw three lines which shall form an equilateral triangle whose side is equal to a given line.

Ex. 942. The feet of the perpendiculars dropped upon the sides of a triangle from any point in the circumference of the circumscribed circle are collinear.



Outline of Proof. The circle having AP as diameter will pass through M and Q. $\therefore \angle 1 = \angle 1'$ and $\angle 2 = \angle 2'$. Similarly

$$\angle 3 = \angle 3',$$

and

$$\angle PKM = \angle PBM.$$

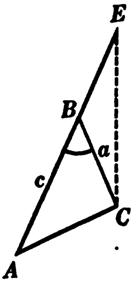
$$\therefore \angle APB = \angle QPK.$$

$$\therefore \angle APQ = \angle BPK.$$

$$\therefore \angle AMQ = \angle BMK.$$

$\therefore$ QM and MK form one str. line.

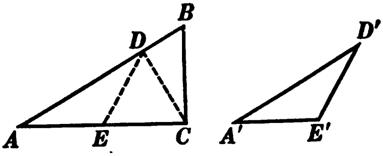
Ex. 943. Given the base, the angle at the vertex, and the sum of the other two sides of a triangle; construct the triangle.



Analysis. Imagine the problem solved and draw $\triangle ABC$. Prolong AB, making $BE = BC$, since the line $c + a$ is given. Since $\angle E = \frac{1}{2} \angle ABC$, $\triangle AEC$ can be constructed.

Ex. 944. The hypotenuse of a right triangle is given in magnitude and position; find the locus of the center of the inscribed circle.

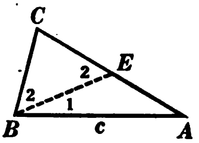
Ex. 945. Prove Prop. VIII, Book IV, by using the following figure, in which $A'D'E'$ is placed in the position ADE.



Ex. 946. Prove Prop. VIII, Book IV, using two triangles such that one will not fall wholly within the other.

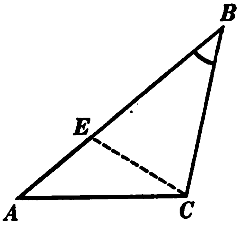
Ex. 947. If two triangles have an angle of one supplementary to an angle of the other, the triangles are to each other as the products of the sides including the supplementary angles. (Prove by method similar to that of Ex. 945.)

Ex. 948. Given base, difference of sides, and difference of base angles; construct the triangle.



Analysis. In the accompanying figure suppose c and EA, $(b-a)$, to be given. A consideration of the figure will show that $\angle 2 = \angle 1 + \angle A$. Add $\angle 1$ to both members of the equation; then $\angle 1 + \angle 2 = 2\angle 1 + \angle A$. But $\angle 1 + \angle 2 = \angle B$. $\therefore \angle B = 2\angle 1 + \angle A$. $\therefore \angle 1 = \frac{1}{2}(\angle B - \angle A)$. The $\triangle BEA$ may now be constructed. The rest of the construction is left for the student.

Ex. 949. Given base, vertex angle, and difference of sides, construct the triangle.

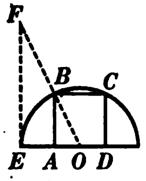


Analysis. $AE = AB - BC$. $\angle AEC = 90° + \frac{1}{2}\angle B$. $\therefore \triangle AEC$ can be constructed.

Ex. 950. If upon the sides of any triangle equilateral triangles are constructed, the lines joining the centers of the equilateral triangles form an equilateral triangle.

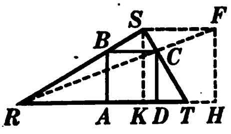
Hint. Circumscribe circles about the three equilateral $\triangle$. Join O, the common point of intersection of the three circles, to A, B, and C, the vertices of the given $\triangle$. Prove each $\angle$ at O the supplement of the $\angle$ opposite in an equilateral $\triangle$, and also the supplement of the $\angle$ opposite in the $\triangle$ to be proved equilateral.

Ex. 951. To inscribe a square in a semicircle.



Analysis. Imagine the problem solved and $ABCD$ the required square. Prove $OA = \frac{AB}{2}$. Draw $EF \perp EO$ meeting OB prolonged at F. $FE : EO = BA : AO$. $\therefore EF = 2\,OE$.

Ex. 952. To inscribe a square in a given triangle.

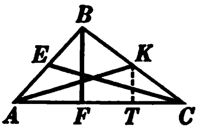


Outline of Solution. Imagine the problem solved and $ABCD$ the required square. Draw $SF \parallel RT$ and construct square $KSFH$. Draw RF, thus determining point C. The cons. will be evident from the figure. To prove $ABCD$ a square, prove $BC = CD$. $BC : SF = CD : FH$. But $SF = FH$. $\therefore BC = CD$.

Ex. 953. In a given square construct a square, having a given side, so that its vertices shall lie in the sides of the given square.

HINT. Construct a rt. △, given the hypotenuse and sum of arms.

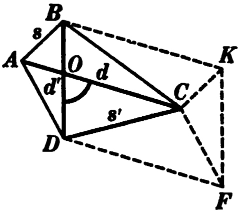
Ex. 954. Construct a triangle, given m_a, m_c, h_b.



ANALYSIS. Imagine the problem solved and that *ABC* is the required △. If *BF* were moved ∥ to itself till it contained *K*, the rt. △ *AKT* would be formed and *KT* would equal $\frac{1}{2}$ *BF*.

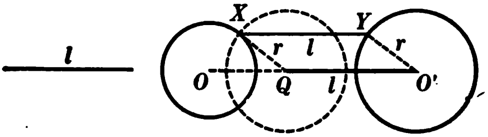
Then △ *AKT* can be made the basis of the required construction.

Ex. 955. Construct a quadrilateral, given two of its opposite sides, its two diagonals, and the angle between them.



OUTLINE OF CONSTRUCTION. Imagine the problem solved and that *ABCD* is the required quadrilateral, s, s', d, d', and ∠ *DOC* being given. By ∥ motion of d and d' the parallelogram *BKFD* may be obtained. The required construction may be begun by drawing ▱ *BKFD*, since two sides and the included ∠ are known. With *K* as center and s as radius, describe an arc; with *D* as center and s' as radius, describe an arc intersecting the first, as at *C*. Construct ▱ *ABKC*, or ▱ *DACF*, to locate *A*.

Ex. 956. Between two circles draw a line which shall be parallel to the line of centers and equal to a given line l.



Ex. 957. Find $x = \frac{abc}{d^2}$, where a, b, c, and d represent given lines.

HINT. Here $x = \frac{ab}{d} \cdot \frac{c}{d}$. Let $\frac{ab}{d} = y$ and construct y. Then construct x.

Ex. 958. Transform any given triangle into an equilateral triangle by a method different from that used in Ex. 865.

ANALYSIS. Call the base of the given triangle b and its altitude a.

Let $x =$ the side of the required equilateral triangle.

Then $\frac{x^2}{4}\sqrt{3} = \frac{1}{2} b \cdot h$. $\therefore \frac{2}{3} b : x = x : h\sqrt{3}$.

BOOK V

REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE

515. Def. A **regular polygon** is one which is both equilateral and equiangular.

Ex. 959. Draw an equilateral triangle. Is it a regular polygon?

Ex. 960. Draw a quadrilateral that is equilateral but not equiangular; equiangular but not equilateral; neither equilateral nor equiangular; both equilateral and equiangular. Which of these quadrilaterals is a regular polygon?

Ex. 961. Find the number of degrees in an angle of a regular dodecagon.

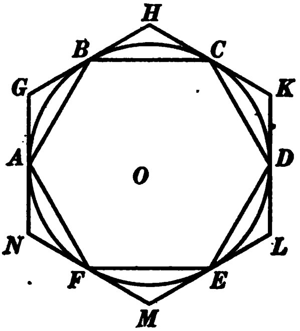
516. Historical Note. The following theorem presupposes the possibility of dividing the circumference into a number of equal arcs. The actual division cannot be obtained by the methods of elementary geometry, except in certain special cases which will be discussed later.

As early as Euclid's time it was known that the angular magnitude about a point (and hence a circumference) could be divided into 2^n, $2^n \cdot 3$, $2^n \cdot 5$, $2^n \cdot 15$ equal angles. In 1796 it was discovered by Gauss, then nineteen years of age, that a regular polygon of 17 sides can be constructed by means of ruler and compasses, and that in general it is possible to construct all polygons having $(2^n + 1)$ sides, n being an integer and $(2^n + 1)$ a prime number. The first four numbers satisfying this condition are 3, 5, 17, 257. Gauss proved also that polygons having a number of sides equal to the product of two or more different numbers of this series can be constructed.

Gauss proved, moreover, that *only a limited class* of regular polygons are constructible by elementary geometry. For a note on the life of Gauss, see § 520.

PROPOSITION I. THEOREM

517. *If the circumference of a circle is divided into any number of equal arcs: (a) the chords joining the points of division form a regular polygon inscribed in the circle; (b) tangents drawn at the points of division form a regular polygon circumscribed about the circle.*



Given circumference ACE divided into equal arcs AB, BC, CD, etc., and let chords AB, BC, CD, etc., join the several points of division, and let the tangents GH, HK, KL, etc., touch the circumference at the several points of division.

To prove $ABCD \cdots$ and $GHKL \cdots$ regular polygons.

ARGUMENT	REASONS
1. $\widehat{AB} = \widehat{BC} = \widehat{CD} = \cdots$.	1. By hyp.
2. $\therefore$ arc $CEA =$ arc $DFB =$ arc $EAC = \cdots$.	2. § 54, 7 *a*.
3. $\therefore \angle ABC = \angle BCD = \angle CDE = \cdots$.	3. § 362, *a*.
4. Also $AB = BC = CD = \cdots$.	4. § 298.
5. $\therefore$ $ABCD \cdots$ is a regular polygon.	5. § 515.
6. Again, $\angle BAG = \angle GBA = \angle CBH = \angle HCB = \cdots$.	6. § 362, *a*.
7. And $AB = BC = CD = \cdots$.	7. Arg. 4.
8. $\therefore \triangle AGB = \triangle BHC = \triangle CKD = \cdots$.	8. § 105.
9 $\therefore \angle G = \angle H = \angle K = \cdots$.	9. § 110.
10. And $AG = GB = BH = \cdots$	10. § 110.

ARGUMENT	REASONS
11. ∴ $GH = HK = KL = \cdots$.	11. § 54, 7 *a*.
12. ∴ $GHKL \cdots$ is a regular polygon. Q.E.D.	12. § 515.

518. Questions. If, in the figure of § 517, the circumference is divided into six equal parts, how many arcs, each equal to arc AB, will arc CEA contain? arc DFB? How many will each contain if the circumference is divided into n equal parts? In step 10, why does $AG = GB$?

519. Cor. *If the vertices of a regular inscribed polygon are joined to the mid-points of the arcs subtended by the sides of the polygon, the joining lines will form a regular inscribed polygon of double the number of sides.*

Ex. 962. An equilateral polygon inscribed in a circle is regular.

Ex. 963. An equiangular polygon circumscribed about a circle is regular.

520. Historical Note. Karl Friedrich Gauss (1777–1855) was born at Brunswick, Germany. Although he was the son of a bricklayer, he was enabled to receive a liberal education, owing to the recognition of his unusual talents by a nobleman. He was sent to the Caroline College but, at the age of fifteen, it was admitted both by professors and pupils that Gauss already knew all that they could teach him. He became a student in the University of Göttingen and while there did some important work on the theory of numbers.

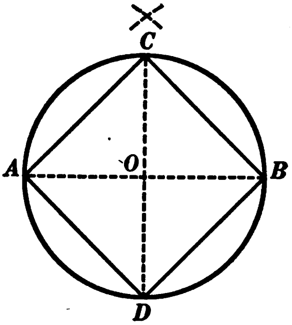
GAUSS

On his return to Brunswick, he lived humbly as a private tutor, until 1807, when he was appointed professor of astronomy and director of the observatory at Göttingen. While there he did important work in physics as well as in astronomy. He also invented the telegraph independently of S. F. B. Morse.

His lectures were unusually clear, and he is said to have given in them the analytic steps by which he developed his proofs; while in his writings there is no hint of the processes by which he discovered his results.

PROPOSITION II. PROBLEM

521. *To inscribe a square in a given circle.*



Given circle O.

To inscribe a square in circle O.

I. The construction is left as an exercise for the student.

II. **Proof**

ARGUMENT	REASONS
1. $AB \perp CD$.	1. By cons.
2. $\therefore \angle COA = 90°$.	2. § 71.
3. $\therefore \widehat{AC} = 90°$, *i.e.* one fourth of the circumference.	3. § 358.
4. $\therefore$ the circumference is divided into four equal parts.	4. Arg. 3.
5. $\therefore$ polygon $ACBD$, formed by joining the points of division, is a square. Q.E.D.	5. § 517, *a*.

III. The discussion is left as an exercise for the student.

522. Cor. *The side of a square inscribed in a circle is equal to the radius multiplied by $\sqrt{2}$; the side of a square circumscribed about a circle is equal to twice the radius.*

Ex. 964. Inscribe a regular octagon in a circle.

Ex. 965. Inscribe in a circle a regular polygon of sixteen sides.

Ex. 966. Circumscribe a square about a circle.

Ex. 967. Circumscribe a regular octagon about a circle.

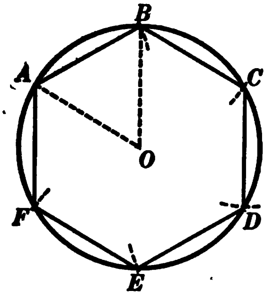
Ex. 968. On a given line as one side, construct a square.

Ex. 969. On a given line as one side, construct a regular octagon.

Ex. 970. If a is the side of a regular octagon inscribed in a circle whose radius is R, then $a = R\sqrt{2-\sqrt{2}}$.

PROPOSITION III. PROBLEM

523. *To inscribe a regular hexagon in a given circle.*



Given circle O.

To inscribe in circle O a regular hexagon.

I. The construction is left as an exercise for the student.

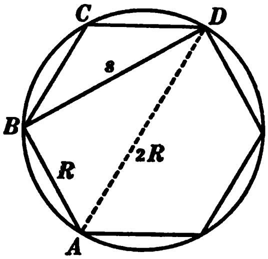
HINT. AB = radius OA.

II. Proof

ARGUMENT	REASONS
1. Draw OB.	1. § 54, 15.
2. Then $\triangle ABO$ is equilateral.	2. By cons.
3. $\therefore \angle O = 60°$.	3. § 213.
4. $\therefore \overset{\frown}{AB} = 60°$, *i.e.* one sixth of the circumference.	4. § 358.
5. $\therefore$ the circumference may be divided into six equal parts.	5. Arg. 4.
6. $\therefore$ polygon $ABCDEF$, formed by joining the points of division, is a regular inscribed hexagon. Q.E.D.	6. § 517, *a*.

III. The discussion is left as an exercise for the student

524. Cor. I. *A regular inscribed triangle is formed by joining the alternate vertices of a regular inscribed hexagon.*



525. Cor. II. *A side of a regular inscribed triangle is equal to the radius of the circle multiplied by $\sqrt{3}$.*

HINT. $\triangle ABD$ is a right triangle whose hypotenuse is $2R$ and one side R.

Ex. 971. Inscribe a regular dodecagon in a circle.

Ex. 972. Divide a given circle into two segments such that any angle inscribed in one segment is five times an angle inscribed in the other.

Ex. 973. Circumscribe an equilateral triangle about a circle.

Ex. 974. Circumscribe a regular hexagon about a circle.

Ex. 975. On a given line as one side, construct a regular hexagon.

Ex. 976. On a given line as one side, construct a regular dodecagon.

Ex. 977. If a is the side of a regular dodecagon inscribed in a circle whose radius is R, then $a = R\sqrt{2-\sqrt{3}}$.

PROPOSITION IV. PROBLEM

526. *To inscribe a regular decagon in a circle.*

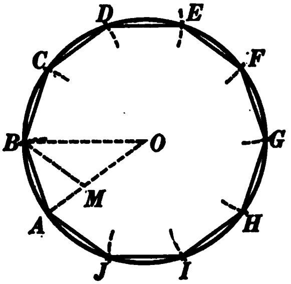


FIG. 1.

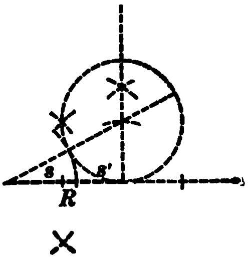


FIG. 2.

Given circle O.

To inscribe in circle O a regular decagon.

I. Construction

1. Divide a radius R, of circle O, in extreme and mean ratio § 465. (See Fig. 2.)

2. In circle O draw a chord AB, equal to s, the greater segment of R.

3. AB is a side of the required decagon.

II. Proof

	Argument		Reasons
1.	Draw radius OA.	1.	§ 54, 15.
2.	On OA lay off $OM = s$.	2.	§ 54, 14.
3.	Draw OB and BM.	3.	§ 54, 15.
4.	Then $OA : OM = OM : MA$.	4.	By cons.
5.	$\therefore \quad OA : AB = AB : MA$.	5.	§ 309.
6.	Also in △s OAB and MAB, $\angle A = \angle A$.	6.	By iden.
7.	$\therefore \triangle OAB \sim \triangle MAB$.	7.	§ 428.
8.	$\because \triangle OAB$ is isosceles, $\triangle MAB$ is isosceles, and $BM = AB$.	8.	§ 94.
9.	But $AB = s = OM$.	9.	By cons.
10.	$\therefore \quad BM = OM$.	10.	§ 54, 1.
11.	$\therefore \triangle BOM$ is isosceles and $\angle MBO = \angle O$.	11.	§ 111.
12.	But $\angle ABM = \angle O$.	12.	§ 424, 1.
13.	$\therefore \angle MBO + \angle ABM$, or $\angle ABO, = 2\angle O$.	13.	§ 54, 2.
14.	$\therefore \angle MAB = 2\angle O$.	14.	§ 111.
15.	In $\triangle ABO$, $\angle ABO + \angle MAB + \angle O = 180°$.	15.	§ 204.
16.	$\therefore 2\angle O + 2\angle O + \angle O$, or $5\angle O, = 180°$.	16.	§ 309.
17.	$\therefore \angle O = 36°$.	17.	§ 54, 8 *a*.
18.	$\therefore \overset{\frown}{AB} = 36°$, *i.e.* one tenth of the circumference.	18.	§ 358.
19.	∴ the circumference may be divided into ten equal parts.	19.	Arg. 18.
20.	∴ polygon $ABCD \cdots$, formed by joining the points of division, is a regular inscribed decagon. Q.E.D.	20.	§ 517, *a*.

III. The discussion is left as an exercise for the student.

527. Cor. I. *A regular pentagon is formed by joining the alternate vertices of a regular inscribed decagon.*

Ex. 978. Construct a regular inscribed polygon of 20 sides.

Ex. 979. The diagonals of a regular inscribed pentagon are equal.

Ex. 980. Construct an angle of 36°; of 72°.

Ex. 981. Divide a right angle into five equal parts.

Ex. 982. The seven diagonals of a regular decagon drawn from any vertex divide the angle at that vertex into eight equal angles.

Ex. 983. Circumscribe a regular pentagon about a circle.

Ex. 984. Circumscribe a regular decagon about a circle.

Ex. 985. On a given line as one side, construct a regular pentagon.

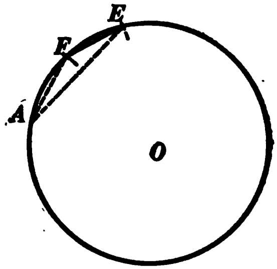
Ex. 986. On a given line as one side, construct a regular decagon.

Ex. 987. The side of a regular inscribed decagon is equal to $\frac{1}{2} R(\sqrt{5}-1)$, where R is the radius of the circle.

Hint. By cons., $R : s = s : R - s$. Solve this proportion for s.

Proposition V. Problem

528. *To inscribe a regular pentedecagon in a circle.*



Given circle O.

To inscribe in circle O a regular pentedecagon.

I. Construction

1. From A, any point in the circumference, lay off chord AE equal to a side of a regular inscribed hexagon. § 523.

2. Also lay off chord AF equal to a side of a regular inscribed decagon. § 526.
3. Draw chord FE.
4. FE is a side of the required pentedecagon.

II. Proof

ARGUMENT	REASONS
1. Arc $AE = \frac{1}{6}$ of the circumference.	1. By cons.
2. Arc $AF = \frac{1}{10}$ of the circumference.	2. By cons.
3. ∴ arc $FE = \frac{1}{6} - \frac{1}{10}$, *i.e.* $\frac{1}{15}$, of the circumference.	3. § 54, 3.
4. ∴ the circumference may be divided into fifteen equal parts.	4. Arg. 3.
5. ∴ the polygon formed by joining the points of division will be a regular inscribed pentedecagon. Q.E.D.	5. § 517, *a*.

III. The discussion is left as an exercise for the student.

529. Note. It has now been shown that a circumference can be divided into the number of equal parts indicated below:

$$\left.\begin{array}{rrrrr} 2, & 4, & 8, & 16, \cdots & 2^n \\ 3, & 6, & 12, & 24, \cdots & 3 \times 2^n \\ 5, & 10, & 20, & 40, \cdots & 5 \times 2^n \\ 15, & 30, & 60, & 120, \cdots & 15 \times 2^n \end{array}\right\} \text{[n being any positive integer].}$$

Ex. 988. Construct an angle of 24°.

Ex. 989. Circumscribe a regular pentedecagon about a given circle.

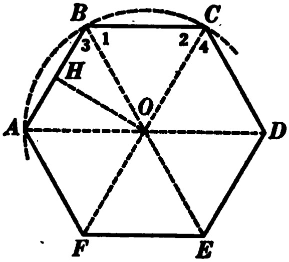
Ex. 990. On a given line as one side, construct a regular pentedecagon.

Ex. 991. Assuming that it is possible to inscribe in a circle a regular polygon of 17 sides, show how it is possible to inscribe a regular polygon of 51 sides.

Ex. 992. If a regular polygon is inscribed in a circle, the tangents drawn at the mid-points of the arcs subtended by the sides of the inscribed polygon form a circumscribed regular polygon whose sides are parallel to the sides of the inscribed polygon, and whose vertices lie on the prolongations of the radii drawn to the vertices of the inscribed polygon.

PROPOSITION VI. THEOREM

530. *A circle may be circumscribed about any regular polygon; and a circle may also be inscribed in it.*



Given regular polygon *ABCD* ····.

To prove: (*a*) that a circle may be circumscribed about it;
(*b*) that a circle may be inscribed in it.

(*a*)

	ARGUMENT		REASONS
1.	Pass a circumference through points *A*, *B*, and *C*.	1.	§ 324.
2.	Connect *O*, the center of the circle, with all the vertices of the polygon.	2.	§ 54, 15.
3.	Then $OB = OC$.	3.	§ 279, *a*.
4.	$\therefore \angle 1 = \angle 2$.	4.	§ 111.
5.	But $\angle ABC = \angle BCD$.	5.	§ 515.
6.	$\therefore \angle 3 = \angle 4$.	6.	§ 54, 3.
7.	Also $AB = CD$.	7.	§ 515.
8.	$\therefore \triangle ABO = \triangle OCD$.	8.	§ 107.
9.	$\therefore OA = OD$ and circumference *ABC* passes through *D*.	9.	§ 110.
10.	In like manner it may be proved that circumference *ABC* passes through each of the vertices of the regular polygon; the circle will then be circumscribed about the polygon. Q.E.D.	10.	By steps similar to 1–9.

(*b*)

Argument	Reasons
1. Again AB, BC, CD, etc., the sides of the given polygon, are chords of the circumscribed circle.	1. § 281.
2. Hence ⊥s from the center of the circle to these chords are equal.	2. § 307.
3. ∴ with O as center, and with a radius equal to one of these ⊥s, as OH, a circle may be described to which all the sides of the polygon will be tangent.	3. § 314.
4. ∴ this circle will be inscribed in the polygon. Q.E.D.	4. § 317.

531. Def. The **center** of a regular polygon is the common center of the circumscribed and inscribed circles; as O, Prop. VI.

532. Def. The **radius** of a regular polygon is the radius of the circumscribed circle, as OA.

533. Def. The **apothem** of a regular polygon is the radius of the inscribed circle, as OH.

534. Def. In a regular polygon the **angle at the center** is the angle between radii of the polygon drawn to the extremities of any side, as $\angle AOF$.

535. Cor. I. *The angle at the center is equal to four right angles divided by the number of sides of the polygon.*

536. Cor. II. *An angle of a regular polygon is the supplement of the angle at the center.*

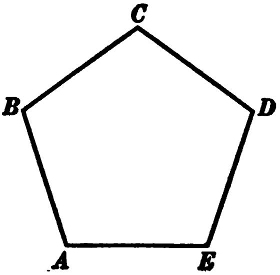
Ex. 993. Find the number of degrees in the angle at the center of a regular octagon. Find the number of degrees in an angle of the octagon

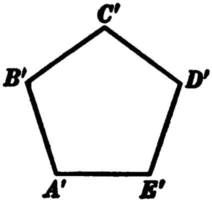
Ex. 994. If the circle circumscribed about a triangle and the circle inscribed in it are concentric, the triangle is equilateral.

Ex. 995. How many sides has a regular polygon whose angle at the center is 30°?

PROPOSITION VII. THEOREM

537. *Regular polygons of the same number of sides are similar.*





Given two regular polygons, $ABCDE$ and $A'B'C'D'E'$, of the same number of sides.

To prove polygon $ABCDE \sim$ polygon $A'B'C'D'E'$.

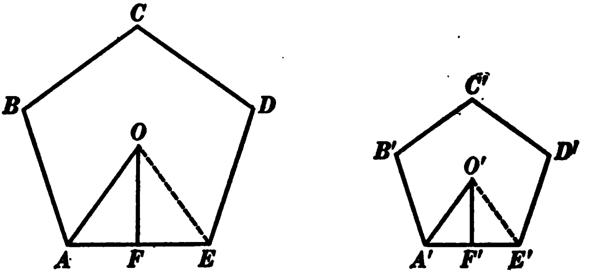
ARGUMENT	REASONS
1. Let n represent the number of sides of each polygon; then each angle of each polygon equals $\dfrac{(n-2)2 \text{ rt. } \angle s}{n}$.	1. § 217.
2. ∴ the polygons are mutually equiangular.	2. Arg. 1.
3. $AB = BC = CD = \cdots$.	3. § 515.
4. $A'B' = B'C' = C'D' = \cdots$	4. § 515.
5. ∴ $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \cdots$.	5. § 54, 8 *a*.
6. ∴ polygon $ABCDE \sim$ polygon $A'B'C'D'E'$. Q.E.D.	6. § 419.

Ex. 996. Two homologous sides of two regular pentagons are 3 inches and 5 inches, respectively; what is the ratio of their perimeters? of their areas?

Ex. 997. The perimeters of two regular hexagons are 30 inches and 72 inches, respectively; what is the ratio of their areas?

PROPOSITION VIII. THEOREM

538. *The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*



Given regular polygons $ABCDE$ and $A'B'C'D'E'$ having the same number of sides. Let OA and $O'A'$ be radii, OF and $O'F'$ apothems, P and P' perimeters, of the two polygons, respectively.

To prove $\frac{P}{P'} = \frac{OA}{O'A'} = \frac{OF}{O'F'}$.

The proof is left as an exercise for the student.

HINT. From §§ 537 and 441, $\frac{P}{P'} = \frac{AE}{A'E'}$. Prove $\triangle AOE \sim \triangle A'O'E'$ (§ 535 and § 428). Then $\frac{AE}{A'E'} = \frac{OA}{O'A'} = \frac{OF}{O'F'}$ (§ 435).

539. Cor. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*

Ex. 998. Two regular hexagons are inscribed in circles whose radii are 7 inches and 8 inches, respectively. Compare their perimeters. Compare their areas.

Ex. 999. The lines joining the mid-points of the radii of a regular pentagon form a regular pentagon whose area is one fourth that of the first pentagon.

MEASUREMENT OF THE CIRCUMFERENCE AND OF THE CIRCLE

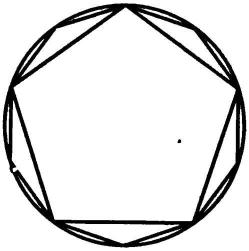
540. The measure of a straight line, *i.e.* its length, is obtained by laying off upon it a straight line taken as a standard or unit (§ 335).

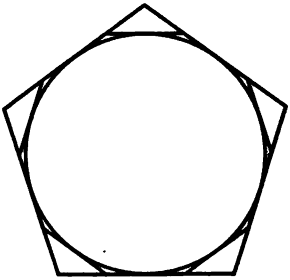
Since a straight line cannot be made to coincide with a curve, it is obvious that some other system of measurement must be adopted for the circumference. The following theorems will develop the principles upon which such measurement is based.

PROPOSITION IX. THEOREM

541. I. *The perimeter and area of a regular polygon inscribed in a circle are less, respectively, than the perimeter and area of the regular inscribed polygon of twice as many sides.*

II. *The perimeter and area of a regular polygon circumscribed about a circle are greater, respectively, than the perimeter and area of the regular circumscribed polygon of twice as many sides.*





The proof is left as an exercise for the student.

HINT. The sum of two sides of a triangle is greater than the third side.

Ex. 1000. A square and a regular octagon are inscribed in a circle whose radius is 10 inches; find:

(*a*) The difference between their perimeters.

(*b*) The difference between their areas.

542. Historical Note. Archimedes (285?–212 B.C.) found the circumference and area of the circle by a method similar to that given in this text. He was born in Syracuse, Sicily, but studied in Egypt at the University of Alexandria.

Although, like Plato, he regarded practical applications of mathematics as of minor importance, yet, on his return to Sicily he is said to have won the admiration of King Hiero by applying his extraordinary mechanical genius to the construction of war-engines with which great havoc was wrought on the Roman army. By means of large lenses and mirrors he is said to have focused the sun's rays and set the Roman ships on fire. Although this story may be untrue, nevertheless such a feat would be by no means impossible.

ARCHIMEDES

Archimedes invented the Archimedes screw, which was used in Egypt to drain the fields after the inundations of the Nile. A ship which was so large that Hiero could not get it launched was moved by a system of cogwheels devised by Archimedes, who remarked in this connection that had he but a fixed fulcrum, he could move the world itself.

The work most prized by Archimedes himself, however, and that which gives him rank among the greatest mathematicians of all time, is his investigation of the mechanics of solids and fluids, his measurement of the circumference and area of the circle, and his work in solid geometry.

Archimedes was killed when Syracuse was captured by the Romans. The story is told that he was drawing diagrams in the sand, as was the custom in those days, when the Roman soldiers came upon him. He begged them not to destroy his circles, but they, not knowing who he was, and thinking that he presumed to command them, killed him with their spears. The Romans, directed by Marcellus, who admired his genius and had given orders that he should be spared, erected a monument to his memory, on which were engraved a sphere inscribed in a cylinder.

The story of the re-discovery of this tomb in 75 B.C. is delightfully told by Cicero, who found it covered with rubbish, when visiting Syracuse.

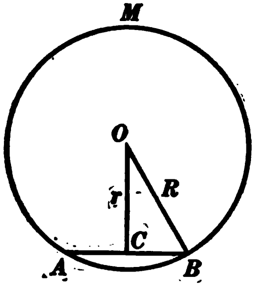
Archimedes is regarded as the greatest mathematician the world has known, with the sole exception of Newton.

Proposition X. Theorem

543. *By repeatedly doubling the number of sides of a regular polygon inscribed in a circle, and making the polygons always regular:*

I. *The apothem can be made to differ from the radius by less than any assigned value.*

II. *The square of the apothem can be made to differ from the square of the radius by less than any assigned value.*



Given AB the side, OC the apothem, and OB the radius of a regular polygon inscribed in circle AMB.

To prove that by repeatedly doubling the number of sides of the polygon:

I. $OB - OC$ can be made less than any assigned value.

II. $\overline{OB}^2 - \overline{OC}^2$ can be made less than any assigned value.

I. Argument	Reasons
1. By repeatedly doubling the number of sides of the inscribed polygon and making the polygons always regular, $\widehat{AB}$, subtended by one side AB of the polygon, can be made less than any previously assigned arc, however small.	1. § 519.
2. ∴ chord AB can be made less than any previously assigned line segment, however small.	2. § 301.

ARGUMENT	REASONS
3. ∴ CB, which is $\frac{1}{2}AB$, can be made less than any previously assigned value, however small.	3. § 544.
4. But $OB - OC < CB$.	4. § 168.
5. ∴ $OB - OC$, being always less than CB, can be made less than any previously assigned value, however small.	5. § 59, 10.
Q.E.D.	

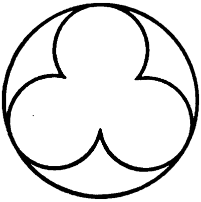
II.

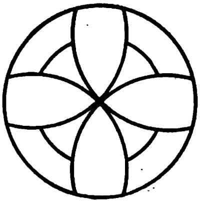
1. Again, $\overline{OB}^2 - \overline{OC}^2 = \overline{CB}^2$.	1. § 447.
2. But CB can be made less than any previously assigned value, however small.	2. I, Arg. 3.
3. ∴ $\overline{CB}^2$ can be made less than any previously assigned value, however small.	3. § 545.
4. ∴ $\overline{OB}^2 - \overline{OC}^2$, being always equal to $\overline{CB}^2$, can be made less than any previously assigned value, however small.	4. § 309.
Q.E.D.	

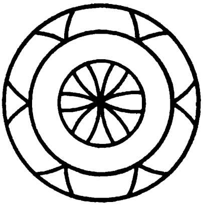
544. *If a variable can be made less than any assigned value, the quotient of the variable by any constant, except zero, can be made less than any assigned value.*

545. *If a variable can be made less than any assigned value, the square of that variable can be made less than any assigned value.* (For proofs of these theorems see Appendix, §§ 586 and 589.)

Ex. 1001. Construct the following designs: (1) on the blackboard, making each line 12 times as long as in the figure; (2) on paper, making each line 4 times as long:





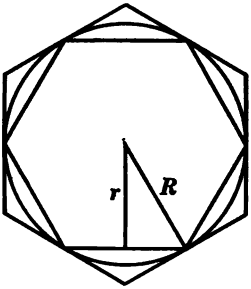


PROPOSITION XI. THEOREM

546. *By repeatedly doubling the number of sides of regular circumscribed and inscribed polygons of the same number of sides, and making the polygons always regular:*

I. *Their perimeters approach a common limit.*

II. *Their areas approach a common limit.*



Given P and p the perimeters, R and r the apothems, and K and k the areas respectively of regular circumscribed and inscribed polygons of the same number of sides.

To prove that by repeatedly doubling the number of sides of the polygons, and making the polygons always regular:

I. P and p approach a common limit.

II. K and k approach a common limit.

I.

ARGUMENT	REASONS
1. Since the two regular polygons have the same number of sides, $\frac{P}{p}=\frac{R}{r}$.	1. § 538.
2. $\therefore \frac{P-p}{P}=\frac{R-r}{R}$.	2. § 399.
3. $\therefore P-p=P\frac{R-r}{R}$.	3. § 54, 7 *a*.
4. But by repeatedly doubling the number of sides of the polygons, and making them always regular, $R-r$ can be	4. § 543, I.

ARGUMENT	REASONS
made less than any previously assigned value, however small.	
5. $\therefore \frac{R-r}{R}$ can be made less than any previously assigned value, however small.	5. § 544.
6. $\therefore P\frac{R-r}{R}$ can be made less than any previously assigned value, however small, P being a decreasing variable.	6. § 547.
7. $\therefore P-p$, being always equal to $P\frac{R-r}{R}$, can be made less than any previously assigned value, however small.	7. § 309.
8. $\therefore P$ and p approach a common limit. Q.E.D.	8. § 548.

II. The proof of II is left as an exercise for the student.

HINT. Since the two regular polygons have the same number of sides, $\frac{K}{k}=\frac{R^2}{r^2}$ (§ 539). The rest of the proof is similar to steps 2–8, § 546, I.

547. *If a variable can be made less than any assigned value, the product of that variable and a decreasing value may be made less than any assigned value.*

548. *If two related variables are such that one is always greater than the other, and if the greater continually decreases while the less continually increases, so that the difference between the two may be made as small as we please, then the two variables have a common limit which lies between them.*

(For proofs of these theorems see Appendix, §§ 587 and 594.)

549. Note. The above proof is limited to *regular* polygons, but it can be shown that the limit of the perimeter of any inscribed (or circumscribed) polygon is the same by whatever method the number of its sides is successively increased, provided that each side approaches zero as a limit.

550. Def. The **length of a circumference** is the common limit which the successive perimeters of inscribed and circumscribed regular polygons (of 3, 4, 5, etc., sides) approach as the number of sides is successively increased and each side approaches zero as a limit.

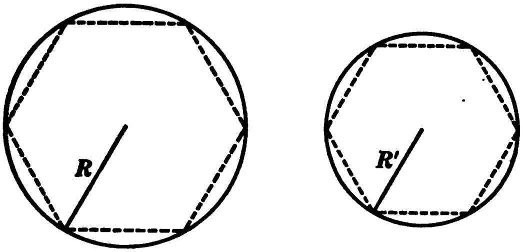
The term "circumference" is frequently used for "the length of a circumference." (See Prop. XII.)

551. The *length of an arc* of a circumference is such a part of the length of the circumference as the central angle which intercepts the arc is of 360°. (See § 360.)

552. The *approximate* length of a circumference is found in elementary geometry by computing the perimeters of a series of regular inscribed and circumscribed polygons which are obtained by repeatedly doubling the number of their sides. The perimeters of these inscribed and circumscribed polygons, since they approach a common limit, may be made to agree to as many decimal places as we please, according to the number of times we double the number of sides of the polygons.

Proposition XII. Theorem

553. *The ratio of the circumference of a circle to its diameter is the same for all circles.*



Given any two circles with circumferences C and C', and with radii R and R', respectively.

To prove $\frac{C}{2R} = \frac{C'}{2R'}$.

ARGUMENT	REASONS
1. Inscribe in the given circles regular polygons of the same number of sides, and call their perimeters P and P'.	1. § 517, *a*.
2. Then $\frac{P}{P'}=\frac{R}{R'}=\frac{2R}{2R'}$.	2. § 538.
3. $\therefore \frac{P}{2R}=\frac{P'}{2R'}$.	3. § 396.
4. As the number of sides of the two regular polygons is repeatedly doubled, P approaches C as a limit, and P' approaches C' as a limit.	4. § 550.
5. $\therefore \frac{P}{2R}$ approaches $\frac{C}{2R}$ as a limit.	5. § 408, *b*.
6. Also $\frac{P'}{2R'}$ approaches $\frac{C'}{2R'}$ as a limit.	6. § 408, *b*.
7. But $\frac{P}{2R}$ is always equal to $\frac{P'}{2R'}$.	7. Arg. 3.
8. $\therefore \frac{C}{2R}=\frac{C'}{2R'}$. Q.E.D.	8. § 355.

554. Def. This constant ratio of the circumference of a circle to its diameter is usually represented by the Greek letter π. It will be shown (§ 568) that its value is approximately $3\frac{1}{7}$; or, more accurately, 3.1416.

555. Cor. I. *The circumference of a circle is equal to* $2\pi R$.

556. Cor. II. *Any two circumferences are to each other as their radii.*

Ex. 1002. If the radius of a wheel is 4 feet, how far does it roll in two revolutions?

Ex. 1003. How many revolutions are made by a wheel whose radius is 3 feet in rolling 44 yards?

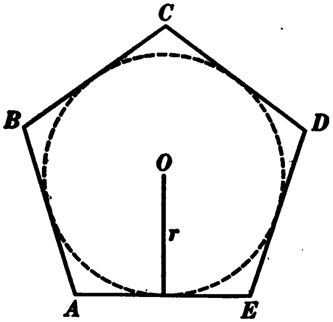
Ex. 1004. (*a*) Find the width of the ring between two concentric circumferences whose lengths are 2 feet and 3 feet, respectively.

(*b*) Assuming the earth's equator to be 25,000 miles, find the width of the ring between it and a concentric circumference 1 foot longer.

(*c*) Write your inference in the form of a general statement.

PROPOSITION XIII. THEOREM

557. *The area of a regular polygon is equal to one half the product of its perimeter and its apothem.*



Given regular polygon $ABCD\cdots$, P its perimeter, and r its apothem.

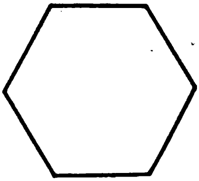
To prove area of $ABCD \cdots = \frac{1}{2} P r$.

ARGUMENT	REASONS
1. In polygon $ABCD\cdots$, inscribe a circle.	1. § 530.
2. Then r, the apothem of regular polygon $ABCD\cdots$, is the radius of circle O.	2. § 533.
3. $\therefore$ area of $ABCD \cdots = \frac{1}{2} P r$. Q.E.D.	3. § 492.

Ex. 1005. Find the area of a regular hexagon whose side is 6 inches.

Ex. 1006. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

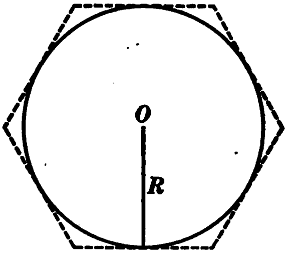
Ex. 1007. The figure represents a flower bed drawn to the scale of 1 inch to 20 feet. Find the number of square feet in the flower bed.



558. Def. The **area of a circle** is the common limit which the successive areas of inscribed and circumscribed regular polygons approach as the number of sides is successively increased and each side approaches zero as a limit.

PROPOSITION XIV. THEOREM

559. *The area of a circle is equal to one half the product of its circumference and its radius.*



Given circle O, with radius R, circumference C, and area K.

To prove $K = \frac{1}{2} CR$.

ARGUMENT	REASONS
1. Circumscribe about circle O a regular polygon. Call its perimeter P and its area S.	1. § 517, *b*.
2. Then $S = \frac{1}{2} PR$.	2. § 557.
3. As the number of sides of the regular circumscribed polygon is repeatedly doubled, P approaches C as a limit.	3. § 550.
4. $\therefore \frac{1}{2} PR$ approaches $\frac{1}{2} CR$ as a limit.	4. § 561.
5. Also S approaches K as a limit.	5. § 558.
6. But S is always equal to $\frac{1}{2} PR$.	6. Arg. 2.
7. $\therefore K = \frac{1}{2} CR$. Q.E.D.	7. § 355.

560. *The product of a variable and a constant is a variable.*

561. *The limit of the product of a variable and a constant, not zero, is the limit of the variable multiplied by the constant.*

(Proofs of these theorems will be found in the Appendix, §§ 585 and 590.)

562. COR. I. *The area of a circle is equal to* πR^2.

HINT. $K = \frac{1}{2} C \cdot R = \frac{1}{2} \cdot 2 \pi R \cdot R = \pi R^2$.

563. Cor. II. *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

564. Cor. III. *The area of a sector whose angle is $a°$ is* $\frac{a}{360}\pi R^2$. (See § 551.)

Ex. 1008. Find the area of a circle whose radius is 8 inches.

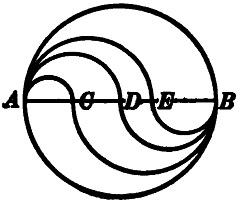
Ex. 1009. Find the area of a sector the angle of which is (a) 45°, (b) 120°, (c) 17°, and the radius, 5 inches. Find the area of the segment corresponding to (b). Hint. Segment = sector − triangle.

Ex. 1010. If the area of one circle is four times that of another, and the radius of the first is 6 inches, what is the radius of the second?

Ex. 1011. Find the area of the ring included between the circumferences of two concentric circles whose radii are 6 inches and 8 inches.

Ex. 1012. Find the radius of a circle whose area is equal to the sum of the areas of two circles with radii 3 and 4, respectively.

Ex. 1013. In the figure the diameter $AB = 2R$, $AC = \frac{1}{4}AB$, $AD = \frac{1}{2}AB$, and E is any point on AB. (1) Find arc AC + arc CB; arc AD + arc DB; arc AE + arc EB. (2) Compare each result with semicircumference AB.



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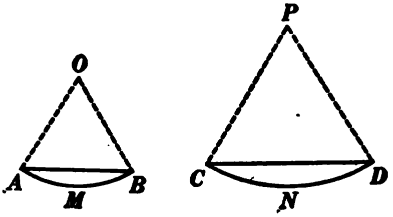
565. Def. In different circles **similar arcs, similar sectors,** and **similar segments** are arcs, sectors, and segments that correspond to equal central angles.

Ex. 1014. Similar arcs are to each other as their radii.

Ex. 1015. Similar sectors are to each other as the squares of their radii.

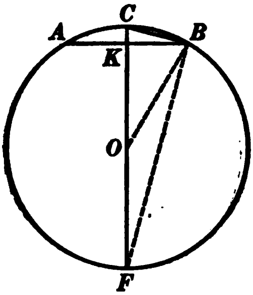
Ex. 1016. Similar segments are to each other as the squares of their radii.

Hint. Prove $\frac{\text{sector } AOB}{\text{sector } CPD} = \frac{\triangle AOB}{\triangle CPD}$. Then apply §§ 396 and 399.



PROPOSITION XV. PROBLEM

566. *Given a circle of unit diameter and the side of a regular inscribed polygon of n sides, to find the side of a regular inscribed polygon of 2n sides.*



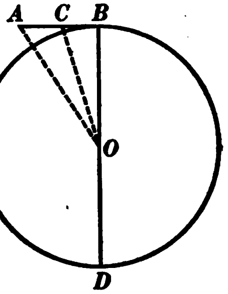
Given circle ABF of unit diameter, AB the side of a regular inscribed polygon of n sides, and CB the side of a regular inscribed polygon of $2n$ sides; denote AB by s and CB by x.

To find x in terms of s.

ARGUMENT	REASONS
1. Draw diameter CF; draw BO and BF.	1. § 54, 15.
2. $\angle CBF$ is a rt. $\angle$.	2. § 367.
3. Also CF is the $\perp$ bisector of AB.	3. § 142.
4. $\therefore \overline{CB}^2 = CF \cdot CK$.	4. § 443, II.
5. Now $CF = 1$, $BO = \frac{1}{2}$, $CO = \frac{1}{2}$.	5. By cons.
6. $\therefore \overline{CB}^2 = x^2 = 1 \cdot CK = CK = CO - KO$ $= \frac{1}{2} - KO$.	6. § 309.
7. $\therefore x^2 = \frac{1}{2} - \sqrt{\overline{BO}^2 - \overline{KB}^2}$ $= \frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{s}{2}\right)^2} = \frac{1 - \sqrt{1 - s^2}}{2}$.	7. § 447.
8. $\therefore x = \sqrt{\frac{1 - \sqrt{1 - s^2}}{2}}$. Q.E.F.	8. § 54, 13.

PROPOSITION XVI. PROBLEM

567. *Given a circle of unit diameter and the side of a regular circumscribed polygon of n sides, to find the side of a regular circumscribed polygon of 2 n sides.*



Given circle O of unit diameter, AB half the side of a regular circumscribed polygon of n sides, and CB half the side of a regular circumscribed polygon of $2n$ sides; denote AB by $\frac{s}{2}$ and CB by $\frac{x}{2}$.

To find x in terms of s.

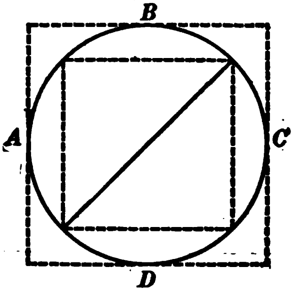
ARGUMENT	REASONS
1. Draw CO and AO.	1. § 54, 15.
2. $\angle BOC = \frac{1}{2}\angle BOA$.	2. § 517, *b*.
3. $\therefore$ in $\triangle OAB$, $AC : CB = AO : BO$.	3. § 432.
4. But $AC = AB - CB$.	4. § 54, 11.
5. And $AO = \sqrt{\overline{AB}^2 + \overline{BO}^2}$.	5. § 446.
6. $\therefore AB - CB : CB = \sqrt{\overline{AB}^2 + \overline{BO}^2} : BO$.	6. § 309.
7. Substituting $\frac{s}{2}$ for AB, $\frac{x}{2}$ for CB, and $\frac{1}{2}$ for BO, $\frac{s}{2} - \frac{x}{2} : \frac{x}{2} = \sqrt{\left(\frac{s}{2}\right)^2 + \left(\frac{1}{2}\right)^2} : \frac{1}{2}$.	7. § 309.
8. $\therefore s - x : x = \sqrt{s^2 + 1} : 1$.	8. § 403.

Argument	Reasons
9. $\therefore s - x = x\sqrt{s^2 + 1}.$	9. § 388.
10. $\therefore x = \dfrac{s}{1 + \sqrt{s^2 + 1}}.$ Q.E.F.	10. Solving for x.

Ex. 1017. Given a circle of unit diameter and an inscribed and a circumscribed square; compute the side of the regular inscribed and the regular circumscribed octagon.

Proposition XVII. Problem

568. *To compute the approximate value of the circumference of a circle in terms of its diameter;* i.e. *to compute the value of* π.



Given circle *ABCD*, with unit diameter.

To compute approximately the circumference of circle *ABCD* in terms of its diameter; *i.e.* to compute the value of π.

Argument	Reasons
1. The ratio of the circumference of a circle to its diameter is the same for all circles.	1. § 553.
2. Since the diameter of the given circle is unity, the side of an inscribed square will be $\frac{1}{2}\sqrt{2}$.	2. § 522.

ARGUMENT	REASONS
3. By using the formula $x=\sqrt{\frac{1-\sqrt{1-s^2}}{2}}$, the sides of regular inscribed polygons of 8, 16, 32, etc., sides may be computed; and by multiplying the length of one side by the number of sides, the length of the perimeter of each polygon may be obtained. The results are given in the table below.	3. § 566.
4. Likewise if the diameter of the given circle is unity, the side of a circumscribed square will be 1.	4. § 522.
5. By using the formula $x=\frac{s}{1+\sqrt{s^2+1}}$, the sides of regular circumscribed polygons of 8, 16, 32, etc., sides may be computed; and by multiplying the length of one side by the number of sides, the length of the perimeter of each polygon may be obtained. The results are given in the following table.	5. § 567.

NUMBER OF SIDES	PERIMETER OF INSCRIBED POLYGON	PERIMETER OF CIRCUMSCRIBED POLYGON
4	2.828427	4.000000
8	3.061467	3.313708
16	3.121445	3.182597
32	3.136548	3.151724
64	3.140331	3.144118
128	3.141277	3.142223
256	3.141513	3.141750
512	3.141572	3.141632
1024	3.141587	3.141602
2048	3.141591	3.141595
4096	3.141592	3.141593

These successive perimeters will be closer and closer approximations of the length of the circumference. By continuing to double the number of sides of the inscribed and circumscribed polygons, perimeters may be obtained which agree to as many orders of decimals as desired.

The last numbers in the table show that the length of a circumference of unit diameter lies between 3.141592 and 3.141593.

$\therefore \pi$, the ratio of any circumference to its diameter, to five decimal places is 3.14159. The value commonly used is 3.1416.

569. Historical Note. The earliest known attempt to find the area of a circle was made by Ahmes, an Egyptian priest, as early as 1700 B.C. His method gave for π the equivalent of 3.1604. His manuscript is preserved in the British Museum.

Archimedes (250 B.C.) gave the value $\frac{22}{7}$, his method being similar to that given in the text.

Hero of Alexandria gave 3 and $3\frac{1}{7}$.

Ptolemy (about 150 B.C.) gave 3.1417.

Metius of Holland (1600 A.D.) gave $\frac{355}{113}$, which is correct to six places.

Lambert (1750 A.D.) proved π an irrational number, and Lindemann (1882) proved it transcendental, *i.e.* not expressible as a root of an algebraic equation.

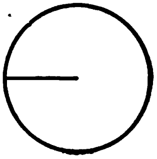
By methods of the calculus the value of π has been computed to several hundred places. Richter carried it to 500 decimal places, and Shanks, in 1873, gave 707 places.

It is impossible to "square the circle," *i.e.* to obtain by accurate construction, with the use of ruler and compasses only, a square equivalent to the area of a circle.

Ex. 1018. Find the area of a circle whose radius is 5 inches. (Let $\pi = 3.1416$.)

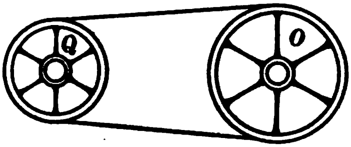
Ex. 1019. The side of an inscribed square is 4 inches. What is the area of the circle?

Ex. 1020. What is the area of a circle inscribed in a square whose side is 6 inches?



Ex. 1021. The figure represents a circular grass plot drawn to the scale of 1 inch to 24 feet. Measure carefully the radius of the circle and find the number of square feet in the grass plot.

Ex. 1022. The figure represents a belt from drive wheel O to wheel Q. The diameter of wheel O is 2 feet and of wheel Q 16 inches. If the drive wheel makes 75 revolutions per minute, how many revolutions per minute will the smaller wheel make?



Ex. 1023. The radius of a circle is 6 inches. What is the area of a segment whose arc is 60°?

Ex. 1024. The radius of a circle is 8 inches. What is the area of the segment subtended by the side of an inscribed equilateral triangle?

Ex. 1025. The diagonals of a rhombus are 16 and 30; find the area of the circle inscribed in the rhombus.

MISCELLANEOUS EXERCISES

Ex. 1026. An equiangular polygon inscribed in a circle is regular if the number of sides is odd.

Ex. 1027. An equilateral polygon circumscribed about a circle is regular if the number of sides is odd.

Ex. 1028. Find the apothem and area in terms of the radius in an equilateral triangle; in a square; in a regular hexagon.

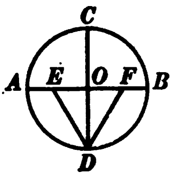
Ex. 1029. The lines joining the mid-points of the radii of a regular pentagon form a regular pentagon. Find the ratio of its area to the area of the original pentagon.

Ex. 1030. Within a circular grass plot of radius 6 feet, a flower bed in the form of an equilateral triangle is inscribed. How many square feet of turf remain?

Ex. 1031. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

Ex. 1032. From a circle of radius 6 is cut a sector whose central angle is 105°. Find the area and perimeter of the sector. $(\pi = \frac{22}{7}.)$

Ex. 1033. Prove that the following method of inscribing a regular pentagon and a regular decagon in a circle is correct. Draw diameter CD perpendicular to diameter AB; bisect OA and join its mid-point to D; take $EF = ED$ and draw FD. FD will be the side of the required pentagon, and OF the side of the required decagon.



Ex. 1034. Divide a given circle into two segments such that any angle inscribed in one segment is twice an angle inscribed in the other; so that an angle inscribed in one segment is three times an angle inscribed in the other; seven times.

Ex. 1035. Show how to cut off the corners of an equilateral triangle so as to leave a regular hexagon; of a square to leave a regular octagon.

Ex. 1036. The diagonals of a regular pentagon form a regular pentagon.

Ex. 1037. The diagonals joining alternate vertices of a regular hexagon form a regular hexagon one third as large as the original one.

Ex. 1038. The area of a regular inscribed octagon is equal to the product of the side of an inscribed square and the diameter.

Ex. 1039. If a is the side of a regular pentagon inscribed in a circle whose radius is R, then $a = \frac{R}{2}\sqrt{10-2\sqrt{5}}$.

Ex. 1040. The area of a regular inscribed dodecagon is equal to three times the square of the radius.

Ex. 1041. Construct an angle of 9°.

Ex. 1042. Construct a regular pentagon, given one of the diagonals.

Ex. 1043. Through a given point construct a line which shall divide a given circumference into two parts in the ratio of 3 to 7; in the ratio of 3 to 5. Can the given point lie within the circle?

Ex. 1044. Transform a given regular octagon into a square.

Ex. 1045. Construct a circumference equal to the sum of two given circumferences.

Ex. 1046. Divide a given circle by concentric circumferences into four equivalent parts.

Ex. 1047. In a given sector whose angle is a right angle inscribe a square.

Ex. 1048. In a given sector inscribe a circle.

Ex. 1049. If two chords of a circle are perpendicular to each other, the sum of the four circles having the four segments as diameters is equivalent to the given circle.

Ex. 1050. The area of a ring between two concentric circumferences whose radii are R and R' respectively is $\pi(R^2 - R'^2)$.

Ex. 1051. The area of the surface between two concentric circles is equal to twice the area of the smaller circle. Find the ratio between their radii.

MISCELLANEOUS EXERCISES ON PLANE GEOMETRY

Ex. 1052. If equilateral triangles are constructed on the sides of any given triangle, the lines joining the vertices of the given triangle to the outer vertices of the opposite equilateral triangles are equal.

Ex. 1053. If, on the arms of a right triangle as diameters, semicircles are drawn so as to lie outside of the triangle, and if, on the hypotenuse as a diameter, a semicircle is drawn passing through the vertex of the right angle, the sum of the areas of the two crescents included between the semicircles is equal to the area of the given triangle.

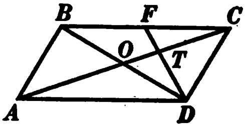
Ex. 1054. The area of the regular inscribed triangle is half that of the regular inscribed hexagon.

Ex. 1055. From a given point draw a secant to a circle such that its internal and external segments shall be equal.

Ex. 1056. Show that the diagonals of any quadrilateral inscribed in a circle divide the quadrilateral into four triangles which are similar, two and two.

Ex. 1057. Through a point P, outside of a circle, construct a secant PAB so that $\overline{AB}^2 = PA \times PB$.

Ex. 1058. The radius of a circle is 6 feet. What are the radii of the circles concentric with it whose circumferences divide its area into three equivalent parts?



Ex. 1059. Given parallelogram $ABCD$, F the mid-point of BC; prove $OT = \frac{1}{2} TC$.

Ex. 1060. Given PT a tangent to a circle at point T, and two other tangents parallel to each other cutting PT at A and B respectively; prove that the radius of the circle is a mean proportional between AT and TB.

Ex. 1061. Show that a mean proportional between two unequal lines is less than half their sum.

Ex. 1062. Given two similar triangles, construct a triangle equivalent to their sum.

Ex. 1063. The square of the side of an inscribed equilateral triangle is equal to the sum of the squares of the sides of the inscribed square and of the inscribed regular hexagon.

Ex. 1064. Prove that the area of a circular ring is equal to the area of a circle whose diameter equals a chord of the outer circumference which is tangent to the inner.

Ex. 1065. If two chords drawn from a common point P on the circumference of a circle are cut by a line parallel to the tangent through P, the chords and the segments of the chords between the two parallel lines are inversely proportional.

Ex. 1066. Construct a segment of a circle similar to two given similar segments and equivalent to their sum.

Ex. 1067. The distance between two parallels is a, and the distance between two points A and B in one parallel is $2b$. Find the radius of the circle which passes through A and B, and is tangent to the other parallel.

Ex. 1068. Tangents are drawn through a point 6 inches from the circumference of a circle whose radius is 9 inches. Find the length of the tangents and also the length of the chord joining the points of contact.

Ex. 1069. If the perimeter of each of the figures, equilateral triangle, square, and circle, is 396 feet, what is the area of each figure?

Ex. 1070. The lengths of two sides of a triangle are 13 and 15 inches, and the altitude on the third side is 12 inches. Find the third side, and also the area of the triangle. (Give one solution only.)

Ex. 1071. If the diameter of a circle is 3 inches, what is the length of an arc of 80°?

Ex. 1072. AD and BC are the parallel sides of a trapezoid $ABCD$, whose diagonals intersect at E. If F is the mid-point of BC, prove that FE prolonged bisects AD.

Ex. 1073. Given a square $ABCD$. Let E be the mid-point of CD, and draw BE. A line is drawn parallel to BE and cutting the square. Let P be the mid-point of the segment of this line within the square. Find the locus of P when the line moves, always remaining parallel to BE. Describe the locus exactly, and prove the correctness of your answer.

Ex. 1074. Let $ABCD$ be any parallelogram, and from any point P in the diagonal AC draw a straight line cutting AB in M, BC in N, CD in L, and AD in K. Prove that $PM \cdot PN = PK \cdot PL$.

Ex. 1075. Find the area of a segment of a circle whose height is 4 inches and chord $8\sqrt{3}$ inches.

Ex. 1076. A square, whose side is 5 inches long, has its corners cut off in such a way as to make it into a regular octagon. Find the area and the perimeter of the octagon.

Ex. 1077. Into what numbers of arcs less than 15 can the circumference of a circle be divided with ruler and compasses only?

Ex. 1078. Through a point A on the circumference of a circle chords are drawn. On each one of these chords a point is taken one third of the distance from A to the other end of the chord. Find the locus of these points, and prove that your answer is correct.

Ex. 1079. In what class of triangles do the altitudes meet within the triangle? on the boundary? outside the triangle? Prove.

Ex. 1080. Given a triangle ABC and a fixed point D on side AC; draw the line through D which divides the triangle into two parts of equal area.

Ex. 1081. The sides of a triangle are 5, 12, 13. Find the radius of the circle whose area is equal to that of the triangle.

Ex. 1082. In a triangle ABC the angle C is a right angle, and the lengths of AC and BC are 5 and 12 respectively; the hypotenuse BA is prolonged through A to a point D so that the length of AD is 4; CA is prolonged through A to E so that the triangles AED and ABC have equal areas. What is the length of AE?

Ex. 1083. Given three points A, B, and C, not in the same straight line; through A draw a straight line such that the distances of B and C from the line shall be equal.

Ex. 1084. Given two straight lines that cut each other; draw four circles of given radius that shall be tangent to both of these lines.

Ex. 1085. Construct two straight lines whose lengths are in the ratio of the areas of two given polygons.

Ex. 1086. The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar circumscribed polygon.

Ex. 1087. Draw a circumference which shall pass through two given points and bisect a given circumference.

Ex. 1088. A parallelogram is constructed having its sides equal and parallel to the diagonals of a given parallelogram. Show that its diagonals are parallel to the sides of the given parallelogram.

HINT. Look for similar triangles.

Ex. 1089. If two chords are divided in the same ratio at their point of intersection, the chords are equal.

Ex. 1090. The sides AB and AC of a triangle ABC are bisected in D and E respectively. Prove that the area of the triangle DBC is twice that of the triangle DEB.

Ex. 1091. Two circles touch externally. How many common tangents have they? Give a construction for the common tangents.

Ex. 1092. Prove that the tangents at the extremities of a chord of a circle are equally inclined to the chord.

Ex. 1093. Two unequal circles touch externally at P; line AB touches the circles at A and B respectively. Prove angle APB a right angle.

Ex. 1094. Find a point within a triangle such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

Ex. 1095. A triangle ABC is inscribed in a circle. The angle B is equal to 50° and the angle C is equal to 60°. What angle does a tangent at A make with BC prolonged to meet it?

Ex. 1096. The bases of a trapezoid are 8 and 12, and the altitude is 6. Find the altitudes of the two triangles formed by prolonging the non-parallel sides until they intersect.

Ex. 1097. The circumferences of two circles intersect in the points A and B. Through A a diameter of each circle is drawn, viz. AC and AD. Prove that the straight line joining C and D passes through B.

Ex. 1098. How many lines can be drawn through a given point in a plane so as to form in each case an isosceles triangle with two given lines in the plane?

Ex. 1099. The lengths of two chords drawn from the same point in the circumference of a circle to the extremities of a diameter are 5 feet and 12 feet respectively. Find the area of the circle.

Ex. 1100. Through a point 21 inches from the center of a circle whose radius is 15 inches a secant is drawn. Find the product of the whole secant and its external segment.

Ex. 1101. The diagonals of a rhombus are 24 feet and 40 feet respectively. Compute its area.

Ex. 1102. On the sides AB, BC, CA of an equilateral triangle ABC measure off segments AD, BE, CF, respectively, each equal to one third the length of a side; draw triangle DEF; prove that the sides of triangle DEF are perpendicular respectively to the sides of triangle ABC.

Ex. 1103. Construct x if (a) $\frac{2}{x}=\frac{x}{3}$; (b) $x=a\sqrt{5}$.

Ex. 1104. Find the area included between a circumference of radius 7 and an inscribed square.

Ex. 1105. What is the locus of the center of a circle of given radius whose circumference cuts at right angles a given circumference?

Ex. 1106. Two chords of a certain circle bisect each other. One of them is 10 inches long; how far is it from the center of the circle?

Ex. 1107. Show how to find on a given straight line of indefinite length a point O which shall be equidistant from two given points A and B in the plane. If A and B lie on a straight line which cuts the given line at an angle of 45° at a point 7 inches distant from A and 17 inches from B, show that OA will be 13 inches.

Ex. 1108. A variable chord passes, when prolonged, through a fixed point outside of a given circle. What is the locus of the mid-point of the chord?

Ex. 1109. A certain parallelogram inscribed in a circle has two sides 20 feet in length and two sides 15 feet in length. What are the lengths of the diagonals?

Ex. 1110. Upon a given base is constructed a triangle one of the base angles of which is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

Ex. 1111. What is the locus of the point of contact of tangents drawn from a fixed point to the different members of a system of concentric circles?

Ex. 1112. Find the locus of all points, the perpendicular distances of which from two intersecting lines are to each other as 3 to 2.

Ex. 1113. The sides of a triangle are a, b, c. Find the lengths of the three medians.

Ex. 1114. Given two triangles; construct a square equivalent to their sum.

Ex. 1115. In a circle whose radius is 10 feet, two parallel chords are drawn, each equal to the radius. Find the area of the portion between these chords.

Ex. 1116. A has a circular garden and B one that is square. The distance around each is the same, namely, 120 rods. Which has the more land, A or B? How much more has he?

Ex. 1117. Prove that the sum of the angles of a pentagram (a five-pointed star) is equal to two right angles.

Ex. 1118. AB and $A'B'$ are any two chords of the outer of two concentric circles; these chords intersect the circumference of the inner circle in points P, Q and P', Q' respectively: prove that $AP \cdot PB = A'P' \cdot P'B'$.

Ex. 1119. A running track consists of two parallel straight portions joined together at the ends by semicircles. The extreme length of the plot inclosed by the track is 176 yards. If the inside line of the track is a quarter of a mile in length, find the cost of seeding this plot at $\frac{1}{4}$ cent a square yard. $(\pi = \frac{22}{7}.)$

Ex. 1120. If two similar triangles, ABC and DEF, have their homologous sides parallel, the lines AD, BE, and CF, which join their homologous vertices, meet in a point.

Ex. 1121. In an acute triangle side $AB = 10$, $AC = 7$, and the projection of AC on AB is 3.4. Construct the triangle and compute the third side BC.

Ex. 1122. Divide the circumference of a circle into three parts that shall be in the ratio of 1 to 2 to 3.

Ex. 1123. The circles having two sides of a triangle as diameters intersect on the third side.

Ex. 1124. Construct a circle equivalent to the sum of two given circles.

Ex. 1125. Assuming that the areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles, prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

Ex. 1126. In a circle of radius 5 a regular hexagon is inscribed. Determine (*a*) the area of one of the segments of the circle which are exterior to the hexagon; (*b*) the area of a triangle whose vertices are three successive vertices of the hexagon; (*c*) the area of the ring bounded by the circumference of the given circle and that of the circle inscribed in the hexagon.

Ex. 1127. Find the locus of the extremities of tangents to a given circle, which have a given length.

Ex. 1128. A ladder rests with one end against a vertical wall and the other end upon a horizontal floor. If the ladder falls by sliding along the floor, what is the locus of its middle point?

Ex. 1129. An angle moves so that its magnitude remains constant and its sides pass through two fixed points. Find the locus of the vertex.

Ex. 1130. The lines joining the feet of the altitudes of a triangle form a triangle whose angles are bisected by the altitudes.

Ex. 1131. Construct a triangle, given the feet of the three altitudes.

Ex. 1132. If the radius of a sector is 2, what is the area of a sector whose central angle is 152°?

Ex. 1133. The rectangle of two lines is a mean proportional between the squares on the lines.

Ex. 1134. Show how to inscribe in a given circle a regular polygon similar to a given regular polygon.

FORMULAS OF PLANE GEOMETRY

570. In addition to the notation given in § 270, the following will be used:

a = side of polygon in general.
b = base of a plane figure.
b, b' = bases of a trapezoid.
C = circumference of a circle.
D = diameter of a circle.
E = sum of exterior angles of a polygon.
h = altitude of a plane figure.
I = sum of interior angles of a polygon.
K = area of a figure in general.
l = line in general.
P = perimeter of polygon in general.
p = projection of b upon a.
R = radius of circle, or radius of regular polygon.
r = apothem of regular polygon, or radius of inscribed circle.
s = the longer of two segments of a line; or
$s = \frac{1}{2}(a + b + c)$.
X = angle in general.
x_a = side of a triangle opposite an acute angle.
x_o = side of a triangle opposite an obtuse angle.

FIGURE	FORMULA	REFERENCE
Any triangle.	$A + B + C = 180°$.	§ 204.
Polygon.	$I = (n - 2)180°$.	§§ 216, 219.
	$E = 4$ rt. $\angle$s.	§ 218.
Central angle.	$X \underline{\propto}$ intercepted arc.	§ 358.
Inscribed angle.	$X \underline{\propto} \frac{1}{2}$ intercepted arc.	§ 365
Angle formed by two chords.	$X \underline{\propto} \frac{1}{2}$ sum of arcs.	§ 377.
Angle formed by tangent and chord.	$X \underline{\propto} \frac{1}{2}$ intercepted arc.	§ 378.
Angle formed by two secants.	$X \underline{\propto} \frac{1}{2}$ difference of arcs.	§ 379.
Angle formed by secant and tangent.	$X \underline{\propto} \frac{1}{2}$ difference of arcs.	§ 379.
Angle formed by two tangents.	$X \underline{\propto} \frac{1}{2}$ difference of arcs.	§ 379
Similar polygons.	$\frac{P}{P'} = \frac{a}{a'}$.	§ 441.
Right triangle.	$c^2 = a^2 + b^2$.	§ 446
Any triangle.	$x_a^2 = a^2 + b^2 - 2ap$.	§ 452
Obtuse triangle.	$x_o^2 = a^2 + b^2 + 2ap$.	§ 455

FIGURE	FORMULA	REFERENCE
Any triangle.	$b^2+c^2=2\left(\frac{a}{2}\right)^2+2\,m_a^2.$	§ 457.
Line divided in extreme and mean ratio.	$l:s=s:l-s.$	§ 465 and Ex. 763.
Rectangle.	$K=b\cdot h.$	§ 475.
Square.	$K=a^2.$	§ 478.
Parallelogram.	$K=b\cdot h.$	§ 481.
Triangle.	$K=\frac{1}{2}\,b\cdot h.$	§ 485.
	$h_a=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$	§ 490.
	$K=\sqrt{s(s-a)(s-b)(s-c)}.$	§ 490.
	$K=\frac{1}{2}(a+b+c)r.$	§ 491.
Polygon.	$K=\frac{1}{2}\,P\cdot r.$	§ 492.
Circle inscribed in triangle.	$r=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$	Ex. 837.
Circle circumscribed about triangle.	$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$	Ex. 838.
Bisector of angle of triangle.	$t_a=\frac{2}{b+c}\sqrt{bcs(s-a)}.$	Ex. 841.
Trapezoid.	$K=\frac{1}{2}(b+b')h.$	§ 495.
Regular polygons of same number of sides.	$\frac{P}{P'}=\frac{R}{R'}=\frac{r}{r'}.$	§ 538.
	$\frac{K}{K'}=\frac{R^2}{R'^2}=\frac{r^2}{r'^2}.$	§ 539.
Circles.	$\frac{C}{2R}=\frac{C'}{2R'}.$	§ 553.
	$C=2\pi R.$	§ 555.
	$\frac{C}{C'}=\frac{R}{R'}.$	§ 556.
Regular polygon.	$K=\frac{1}{2}\,P\cdot r.$	§ 557.
Circle.	$K=\frac{1}{2}\,C\cdot R.$	§ 559.
	$K=\pi R^2.$	§ 562.
Circles.	$\frac{K}{K'}=\frac{R^2}{R'^2}=\frac{D^2}{D'^2}.$	§ 563.
Sector.	$K=\frac{\text{central}\angle}{360^\circ}\pi R^2.$	§ 564.
Segment.	$K=\text{sector}\mp\text{triangle}.$	Ex. 1009

APPENDIX TO PLANE GEOMETRY

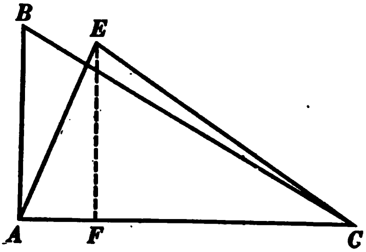
MAXIMA AND MINIMA

571. Def. Of all geometric magnitudes that satisfy given conditions, the greatest is called the **maximum**, and the least is called the **minimum.***

572. Def. Isoperimetric figures are figures which have the same perimeter.

PROPOSITION I. THEOREM

573. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Given ▵ ABC and AEC, with AB and AC equal to AE and AC respectively. Let $\angle CAB$ be a rt. $\angle$ and $\angle CAE$ an oblique $\angle$.

To prove $\triangle ABC > \triangle AEC$.

Draw the altitude EF.

▵ ABC and AEC have the same base, AC.

Altitude $AB >$ altitude EF.

$\therefore \triangle ABC > \triangle AEC$. Q.E.D.

574. Cor. I. *Conversely, if two sides are given, and if the triangle is a maximum, then the given sides include a right angle.*

HINT. Prove by *reductio ad absurdum*.

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* In later mathematics a somewhat broader use will be made of these terms.

575. Cor. II. *Of all parallelograms having given sides, the one that is rectangular is a maximum, and conversely.*

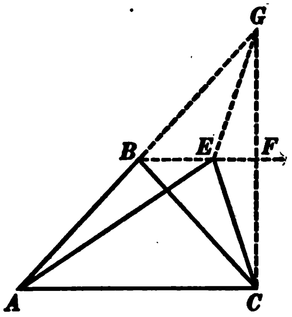
Ex. 1135. Construct the maximum parallelogram having two lines of given lengths as diagonals.

Ex. 1136. What is the minimum line from a given point to a given line?

Ex. 1137. Of all triangles having the same base and altitude, that which is isosceles has the minimum perimeter.

PROPOSITION II. THEOREM

576. *Of all equivalent triangles having the same base, that which is isosceles has the least perimeter.*



Given equivalent △ ABC and AEC with the same base AC, and let $AB = BC$ and $AE \neq EC$.

To prove $AB + BC + CA < AE + EC + CA$.

Draw $CF \perp AC$ and let CF meet the prolongation of AB at G.

Draw EG and BE and prolong BE to meet GC at F.

$BF \parallel AC$.

$\angle CBF = \angle FBG$.

BF bisects CG and is $\perp CG$.

$\therefore BC = BG$ and $EC = EG$.

$$AB + BG < AE + EG.$$

$$\therefore AB + BC < AE + EC.$$

$\therefore AB + BC + CA < AE + EC + CA.$ Q.E.D.

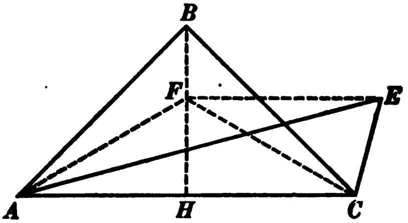
Ex. 1138. Of all equivalent triangles having the same base, that which has the least perimeter is isosceles (Prove by *reductio ad absurdum.*)

Ex. 1139. Of all equivalent triangles, the one that has the minimum perimeter is equilateral.

Ex. 1140. State and prove the converse of Ex. 1139.

PROPOSITION III. THEOREM

577. *Of all isoperimetric triangles on the same base, the isosceles triangle is the maximum.*



Given isosceles △ ABC and any other △ as AEC having the same base and the same perimeter as △ ABC.

To prove △ $ABC >$ △ AEC.

Draw $BH \perp AC$, EF from $E \parallel AC$, and draw AF and FC.

△ AFC is isosceles.

∴ perimeter of △ $AFC <$ perimeter of △ AEC.

∵ perimeter of △ $AFC <$ perimeter of △ ABC.

∴ $AF < AB$.

∴ $FH < BH$.

∴ △ $AFC <$ △ ABC.

∴ △ $AEC <$ △ ABC; *i.e.* △ $ABC >$ △ AEC. Q.E.D

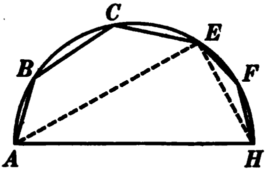
Ex. 1141. Of all triangles having a given perimeter and a given base the one that has the maximum area is isosceles.

Ex. 1142. What is the maximum chord of a circle? What is the maximum and what the minimum line that can be drawn from a given exterior point to a given circumference?

Ex. 1143. Of all triangles having a given perimeter, the one that has the maximum area is equilateral.

PROPOSITION IV. THEOREM

578. *Of all polygons having all their sides but one equal, respectively, to given lines taken in order, the maximum can be inscribed in a semicircle having the undetermined side as diameter.*



Given polygon $ABCEFH$, the maximum of all polygons subject to the condition that AB, BC, CE, EF, FH, are equal respectively to given lines taken in order.

To prove that the semicircumference described with AH as diameter passes through B, C, E, and F.

Suppose that the semicircumference with AH as diameter does not pass through some vertex, as E. Draw AE and EH.

Then $\angle AEH$ is not a rt. $\angle$.

Then if the figures $ABCE$ and EFH are revolved about E until AEH becomes a rt. $\angle$, $\triangle AEH$ will be increased in area.

$\therefore$ polygon $ABCEFH$ can be increased in area without changing any of the given sides.

But this contradicts the hypothesis that polygon $ABCEFH$ is a maximum.

$\therefore$ the supposition that vertex E is not on the semicircumference is false.

$\therefore$ the semicircumference passes through E.

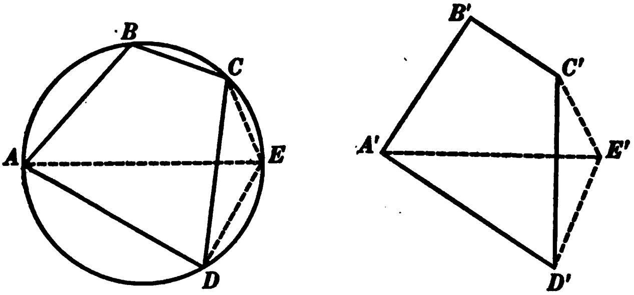
In the same way it may be proved that every vertex of the polygon lies on the semicircumference. Q.E.D.

Ex. 1144. Given the base and the vertex angle of a triangle, construct the triangle so that its area shall be a maximum.

Ex. 1145. Find the point in a given straight line such that the tangents drawn from it to a given circle contain a maximum angle.

PROPOSITION V. THEOREM

579. *Of all polygons that have their sides equal, respectively, to given lines taken in order, the polygon that can be circumscribed by a circle is a maximum.*



Given polygon $ABCD$ which is circumscribed by a ⊙, and polygon $A'B'C'D'$ which cannot be circumscribed by a ⊙, with $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, and $DA = D'A'$.

To prove $ABCD > A'B'C'D'$.

From any vertex as A draw diameter AE; draw EC and ED. On $C'D'$, which equals CD, construct $\triangle D'C'E'$ equal to $\triangle DCE$; draw $A'E'$.

The circle whose diameter is $A'E'$ does not pass through all the points B', C', D'. (Hyp.)

∴ either $ABCE$ or EDA or both must be greater, and neither can be less, than the corresponding part of polygon $A'B'C'E'D'$ (§ 578).

∴ $ABCED > A'B'C'E'D'$. But $\triangle DCE = \triangle D'C'E'$.

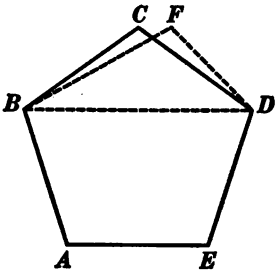
∴ $ABCD > A'B'C'D'$. Q.E.D.

Ex. 1146. In a given semicircle inscribe a trapezoid whose area is a maximum.

Ex. 1147. Of all equilateral polygons having a given side and a given number of sides, the one that is regular is a maximum.

Proposition VI. Theorem

580. *Of all isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Given polygon $ABCDE$ the maximum of all isoperimetric polygons of the same number of sides.

To prove $AB = BC = CD = DE = EA$.

Suppose, if possible, $BC > CD$.

On BD as base construct an isosceles $\triangle BFD$ isoperimetric with $\triangle BCD$.

$$\triangle BFD > \triangle BCD.$$

$\therefore$ polygon $ABFDE >$ polygon $ABCDE$.

But this contradicts the hypothesis that $ABCDE$ is the maximum of all isoperimetric polygons having the same number of sides.

$$\therefore BC = CD.$$

In like manner any two adjacent sides may be proved equal

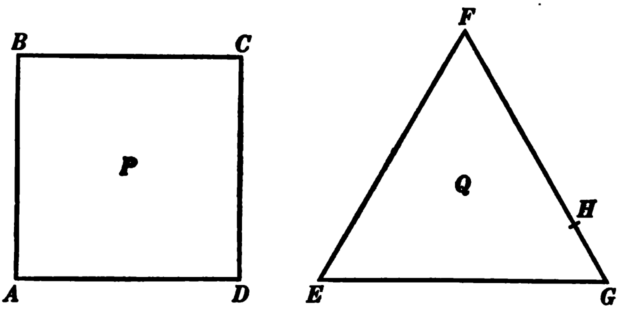
$$\therefore AB = BC = CD = DE = EA.$$ Q.E.D.

581. Cor. *Of all isoperimetric polygons of the same number of sides, the maximum is regular.*

Ex. 1148. In a given segment inscribe a triangle whose perimeter is a maximum.

PROPOSITION VII. THEOREM

582. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Given the isoperimetric polygons P and Q, and let P have one more side than Q.

To prove $P > Q$.

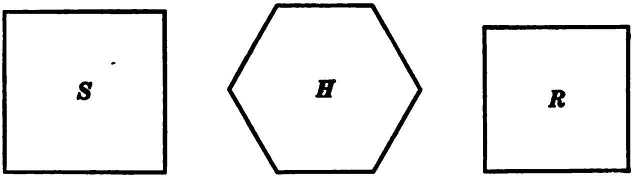
In one side of Q take any point as H.

$EFHG$ may be considered as an irregular polygon having the same number of sides as P.

$\therefore P > EFHG$ · i.e. $P > Q$. Q.E.D

PROPOSITION VIII. THEOREM

583. *Of two equivalent regular polygons, that which has the greater number of sides has the smaller perimeter.*



Given square $S \Bumpeq$ regular hexagon H.

To prove perimeter of $S >$ perimeter of H.

Construct square R isoperimetric with H.

Area of $H >$ area of R; *i.e.* area of $S >$ area of R.
$\therefore$ perimeter of $S >$ perimeter of R.
$\therefore$ perimeter of $S >$ perimeter of H. Q.E.D

584. Cor. *Of all polygons having a given number of sides and a given area, that which has a minimum perimeter is regular.*

Ex. 1149. Among the triangles inscribed in a given circle, the one that has a maximum perimeter is equilateral.

Ex. 1150. Of all polygons having a given number of sides and inscribed in a given circle, the one that has a maximum perimeter is regular.

VARIABLES AND LIMITS. THEOREMS

Proposition I. Theorem

585. *If a variable can be made less than any assigned value, the product of the variable and any constant can be made less than any assigned value.*

Given a variable V, which can be made less than any previously assigned value, however small, and let K be any constant.

To prove that $V \cdot K$ may be made as small as we please, *i.e.* less than any assigned value.

Assign any value, as a, no matter how small.

Now a value for V may be found as small as we please.

Take $V < \frac{a}{K}$. Then $V \cdot K < a$; *i.e.* $V \cdot K$ may be made less than any assigned value. Q.E.D.

586. Cor. I. *If a variable can be made less than any assigned value, the quotient of the variable by any constant, except zero, can be made less than any assigned value.*

Hint. $\frac{V}{K} = \frac{1}{K} \cdot V$, which is the product of the variable and a constant.

587. Cor. II. *If a variable can be made less than any assigned value, the product of that variable and a de-*

creasing value may be made less than any assigned value.

HINT. Apply the preceding theorem, using as K a value greater than any value of the decreasing multiplier.

588. Cor. III. *The product of a variable and a variable may be a constant or a variable.*

589. Cor. IV. *If a variable can be made less than any assigned value, the square of that variable can be made less than any assigned value.* (Apply Cor. II.)

Ex. 1151. Which of the corollaries under Prop. I is illustrated by the theorem: "The product of the segments of a chord drawn through a fixed point within a circle is constant"?

PROPOSITION II. THEOREM

590. *The limit of the product of a variable and a constant, not zero, is the limit of the variable multiplied by the constant.*

Given any variable V which approaches the finite limit L, and let K be any constant not zero.

To prove the limit of $K \cdot V = K \cdot L$.

Let $R = L - V$; then $V = L - R$.

$\therefore K \cdot V = K \cdot L - K \cdot R$.

But the limit of $K \cdot R = 0$.

$\therefore$ the limit of $K \cdot V =$ the limit of $(K \cdot L - K \cdot R) = K \cdot L$.

Q.E.D.

591. Cor. *The limit of the quotient of a variable by a constant is the limit of the variable divided by the constant.*

HINT. $\frac{V}{K} = \frac{1}{K} \cdot V$, which is the product of the variable and a constant.

PROPOSITION III. THEOREM

592. *If two variables approach finite limits, not zero, then the limit of their product is equal to the product of their limits.*

Given variables V and V' which approach the finite limits L and L', respectively.

To prove the limit of $V \cdot V' = L \cdot L'$.

Let $R = L - V$ and $R' = L' - V'$.

Then $V = L - R$ and $V' = L' - R'$.

$\therefore V \cdot V' = L \cdot L' - (L' \cdot R + L \cdot R' - R \cdot R')$.

But the limit of $(L' \cdot R + L \cdot R' - R \cdot R') = 0$.

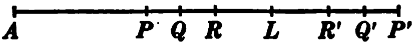
$\therefore$ the limit of $V \cdot V' =$ the limit of $[L \cdot L' - (L' \cdot R + L \cdot R' - R \cdot R')] = L \cdot L'$.

$\therefore$ the limit of $V \cdot V' = L \cdot L'$. Q.E.D.

593. Cor. *If each of any finite number of variables approaches a finite limit, not zero, then the limit of their product is equal to the product of their limits.*

PROPOSITION IV. THEOREM

594. *If two related variables are such that one is always greater than the other, and if the greater continually decreases while the less continually increases, so that the difference between the two may be made as small as we please, then the two variables have a common limit which lies between them.*



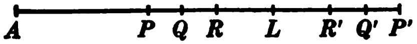
Given the two related variables AP and AP', AP' greater than AP, and let AP and AP' be such that as AP increases AP' shall decrease, so that the difference between AP and AP' shall approach zero as a limit.

To prove that AP and AP' have a common limit, as AL, which lies between AP and AP'.

Denote successive values of AP by AQ, AR, etc., and denote the corresponding values of AP' by AQ', AR', etc.

Since every value which AP assumes is less than any value which AP' assumes (Hyp.) $\therefore AP < AR'$.

But AP is continually increasing.



Hence AP has some limit. (By def. of a limit, § 349.)

Since any value which AP' assumes is greater than every value which AP assumes (Hyp.) $\therefore AP' > AR$.

But AP' is continually decreasing.

Hence AP' has some limit. (By def. of a limit, § 349.)

Suppose the limit of $AP \neq$ the limit of AP'.

Then let the limit of AP be AK, while that of AP' is AK'. Then AK and AK' have some finite values, as m and m', and their difference is a finite value, as d.

But the difference between some value of AP and the corresponding value of AP' cannot be less than the difference of the two limits AK and AK'.

This contradicts the hypothesis that the difference between AP and AP' shall approach zero as a limit.

$\therefore$ the limit of $AP =$ the limit of AP' and lies between AP and AP as AL. Q.E.D.

595. THEOREM. *With every straight line segment there is associated a number which may be called its measure-number.*

For line segments commensurable with the unit this theorem was considered in §§ 335 and 336; we shall now consider the case where the segment is incommensurable with the chosen unit.

Given the straight line segment a and the unit segment u; to express a in terms of u.

Apply u (as a measure) to a as many times as possible, suppose t times, then

$$t \cdot u < a < (t+1)\,u.$$

Now apply some fractional part of u, say $\frac{u}{p}$, to a, and suppose it is contained t_1 times, then

$$\frac{t_1}{p}\cdot u < a < \frac{t_1+1}{p}\cdot u.$$

Then apply smaller and smaller fractional parts of u to a, say $\frac{u}{p^2}, \frac{u}{p^3}, \frac{u}{p^4}, \cdots$, and suppose them to be contained $t_2, t_3, t_4 \cdots$ times respectively, then

$$\frac{t_2}{p^2}\cdot u < a < \frac{t_2+1}{p^2}\cdot u, \qquad \frac{t_3}{p^3}\cdot u < a < \frac{t_3+1}{p^3}\cdot u, \qquad \cdots.$$

Now the infinite series of increasing numbers $t, \frac{t_1}{p}, \frac{t_2}{p^2}, \cdots$ none of which exceeds the finite number $t+1$, defines a number n (the limit of this series) which we shall call the measure-number of a with respect to u. Moreover, this number n is unique, *i.e.* independent of p (the number of parts into which the unit was divided), for if m is any number such that $m < n$, then $m \cdot u < a$, and if $m > n$, then $m \cdot u > a$; we are therefore justified in associating the number n with a, and in saying that $n \cdot u = a$.

596. Note. Manifestly, the above procedure may be applied to any geometric magnitude whatever, *i.e.* every geometric magnitude has a unique measure-number.

597. Cor. *If a magnitude is variable and approaches a limit, then, as the magnitude varies, the successive measure-numbers of the variable approach as their limit the measure-number of the limit of the magnitude.*

598. Discussion of the problem:

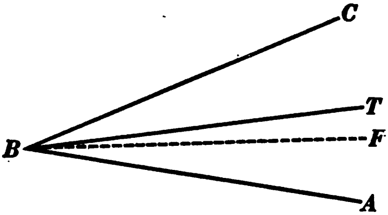
To determine whether two given lines are commensurable or not; and if they are commensurable, to find their common measure and their ratio (§ 345).

Moreover, GD is the greatest common measure of AB and CD. For every measure of AB is a measure of its multiple CE. Hence, every common measure of AB and CD is a common measure of CE and CD and therefore a measure of their differ ence *ED*, and therefore of *AF*, which is a multiple of *ED*. Hence, every common measure of *AB* and *CD* is a common measure of *AB* and *AF* and therefore a measure of their difference *FB*. Again, every common measure of *ED* and *FB* is a common measure of *ED* and *EG* (a multiple of *FB*) and hence of their difference *GD*. Hence, no common measure of *AB* and *CD* can exceed *GD*. Therefore, *GD* is the *greatest* common measure of *AB* and *CD*.

Now, if *AB* and *CD* are *commensurable*, the process must terminate; for any common measure of *AB* and *CD* is a measure of each remainder, and every segment applied as a measure is less than the preceding remainder. Now, if the process did not terminate, a remainder could be reached which would be less than any assigned value, however small, and therefore less than the greatest common measure, which is absurd.

If *AB* and *CD* are *incommensurable*, the process will not terminate; for, if it did, the last remainder obtained would be a common measure of *AB* and *CD*, as shown above.

599. THEOREM. *An angle can be bisected by only one line.*



Given $\angle ABC$, bisected by *BT*.

To prove that no other bisector of $\angle ABC$ exists.

Suppose that another bisector of $\angle ABC$ exists, *e.g.* *BF*.

Then $\angle ABF = \angle ABT$. This is impossible.

$\therefore$ no other bisector of $\angle ABC$ exists. Q.E.D

600. Note on Axioms. The thirteen axioms (§ 54) refer to *numbers* and may be used when referring to the *measure-numbers* of geometric magnitudes. Axioms 2–9 are not applicable always to *equal figures*. (See Exs. 800 and 801.) Axioms 7 and 8 hold for positive numbers only, but do not hold for negative numbers, for zero, nor for infinity; axioms 11 and 12 hold only when the number of parts is finite.

601. The methods of proving locus theorems are illustrated by the following proofs of § 141 and § 305.

(a) *The locus of all points equidistant from the ends of a given line is the perpendicular bisector of that line.*

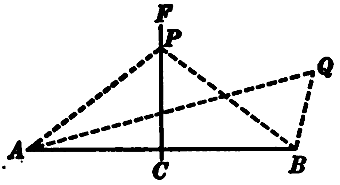


FIG. 1.

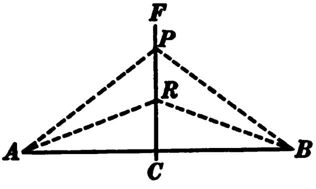


FIG. 2.

Given line AB and its ⊥ bisector, CF.

To prove CF the locus of all points equidistant from A and B.

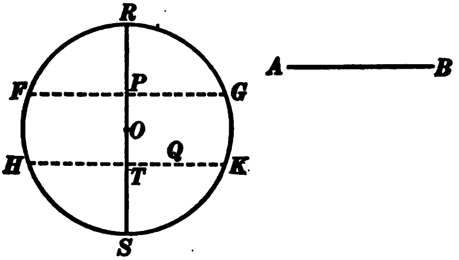
First Method (Fig. 1)

ARGUMENT	REASONS
1. Let P be any point in CF. Then P is equidistant from A and B, *i.e.* every point in CF satisfies the prescribed condition.	1. § 134.
2. Let Q be any point not in CF. Then Q is unequally distant from A and B, *i.e.* no point outside of CF satisfies the prescribed condition.	2. § 140.
3. ∴ CF is the locus of all points equidistant from A and B. Q.E.D.	3. § 130.

Second Method (Fig. 2)

ARGUMENT	REASONS
1. Same as Arg. 1, above.	
2. Let R be any point such that $RA = RB$. Then R lies in CF, *i.e.* every point which satisfies the prescribed condition lies in CF.	2. § 139.
3. Same as Arg. 3, above.	

(*b*) *The locus of the mid-points of all chords of a circle parallel to a given line is the diameter perpendicular to the line.*



Given circle O, line AB, and diameter $RS \perp AB$.

To prove RS the locus of the mid-points of all chords of circle O that are $\parallel AB$.

ARGUMENT	REASONS
1. Let P be any point in diameter RS. Through P draw $FG \parallel AB$.	1. § 179.
2. Now $RS \perp AB$.	2. By hyp.
3. $\therefore RS \perp FG$.	3. § 193.
4. $\therefore P$ is the mid-point of FG, a chord $\parallel AB$.	4. § 302.
5. Let Q be any point not in diameter RS. Through Q draw $HK \parallel AB$, intersecting RS in T.	5. § 179.
6. Then $RS \perp HK$.	6. § 193.
7. $\therefore T$ is the mid-point of HK, *i.e.* Q is not the mid-point of HK, a chord $\parallel AB$.	7. § 302.
8. $\therefore RS$ is the locus of the mid-points of all chords of circle O that are $\parallel AB$.	8. § 130.
Q.E.D.	

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(313)

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MUCH experience has taught that if agriculture is made the object of brain work, as well as of manual labor, better farms, and better farmers, will be produced. This book, therefore, is intended to present in a concise, practical manner for pupils in elementary schools, those fundamental principles upon which successful agriculture depends.

¶ The farm is treated as the center of interest, and its industries, economies, and science are discussed at some length. The book has been prepared with special reference both to simplicity and to scientific accuracy, and is based on the observation of the every-day facts of rural life, and on a system of simple experiments well within the resources of any school.

¶ The pupil is taught the reasons for the more important agricultural operations, and the explanations of the phenomena which accompany them. The soil and vegetation are first taken up, including such important topics as the rotation of crops, parasites of plants, seed testing, animals that destroy insects, and the improvement of plants. Then follow chapters on dairying, live stock, poultry, bee-keeping, and the improvement of home and school yards.

¶ At the end of each chapter is a summary of what has been presented, furnishing in concise form definite statements for the pupil to learn, and supplying to the teacher a basis for drill work. An extensive appendix contains, among other useful information, fodder tables, a table of fertilizing constituents in feeding stuffs, and sections on milk testing, silage, contents of fields and lots, and quantities of seed required to the acre.

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FOR RURAL EDUCATION

BEXELL AND NICHOLS'S PRINCIPLES OF BOOKKEEPING AND FARM ACCOUNTS

Textbook, Blanks, Reference Book,

The first book in its field to teach a simple system of keeping such accounts as are of value to farmers. It deals only with the commodities and conditions of farming. There are many helpful exercises and review questions. All the material used has been tested both in the classroom and on the farm. The course is adapted for grammar or high schools and may be conducted by teachers having little experience in accounting.

BRACE AND MAYNE'S FARM SHOP WORK

A course for rural elementary and secondary schools, which has a distinct practical value in that it teaches boys how to make articles that are of great usefulness on the farm. The fact that these products must stand the test of use arouses greater interest in the pupil and causes him to do more careful work. The course does not require expensive shop equipment or costly raw materials. The exercises provide instruction and training in woodworking, blacksmithing, cement and concrete work, and harness mending. The various tools needed, their use and care, are fully described.

BRICKER'S AGRICULTURAL EDUCATION FOR TEACHERS . . . .

This book discusses the chief lines of development in agricultural teaching and by means of clear directions attempts to vitalize and systematize the work. It aims to give the teacher the true spirit of "rural-mindedness", and to supply him with correct professional knowledge of what to teach and how and when to teach it. The book will be of value to agricultural and normal schools, active and prospective teachers, district and county superintendents, supervisors of rural education, and teachers' reading circles.

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